# Tuesday, August 28, 2012 (Day 2)

Discussing Limits (Section 3-1 in book)

Continuing graphical approach that we started yesterday

Re-cast the definition of limit as a 3-part test.

For $\lim_{x \to c} f(x)$ to exist, the function must pass this 3-part test:

(a) the left-sided limit $\lim_{x \to c^-} f(x)$ must exist

(b) the right-sided limit $\lim_{x \to c^+} f(x)$ must exist

(c) the values of the limits in (a) + (b) must match

Yesterday's examples were all of the form

graph of $f$ $\longrightarrow$ description of limit behavior of $f$.

Now do a different kind of example.

description of limit behavior of $f$ $\longrightarrow$ Sketch a graph of $f$

<u>Example</u> Sketch a graph that satisfies all these conditions:

$$f(1) = 3$$

$$\lim_{x \to 1^-} = 4$$

$$\lim_{x \to 1^+} = 2$$

(this is like exercise 3-1 #40)

## Solution

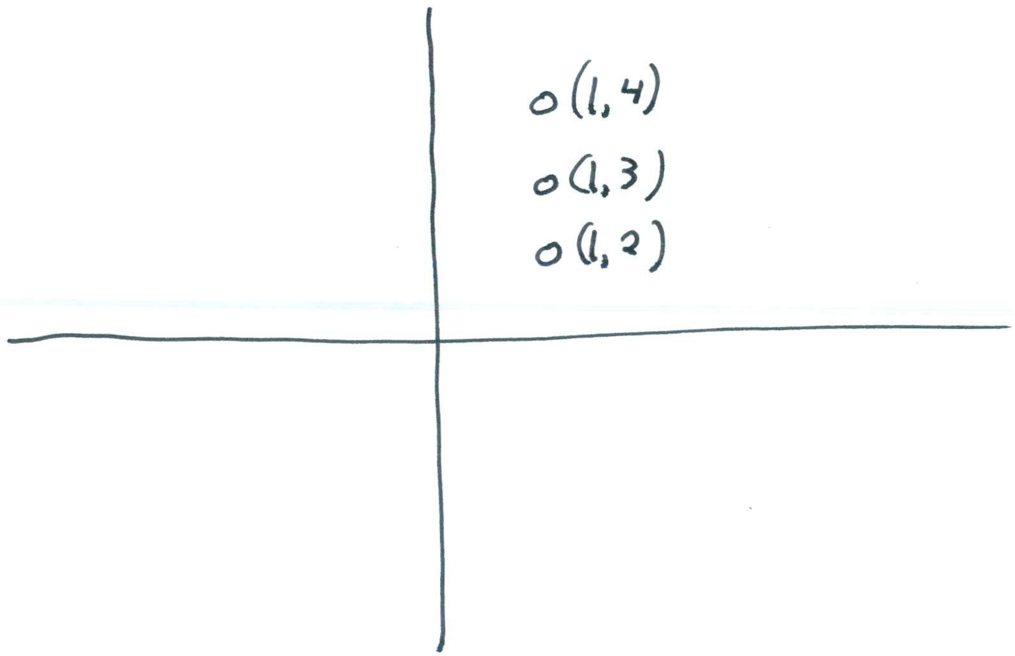
~~Rec~~ Identify the locations that are mentioned in the problem.

$$(x,y) = (1,3)$$

$$(x,y) = (1,4)$$

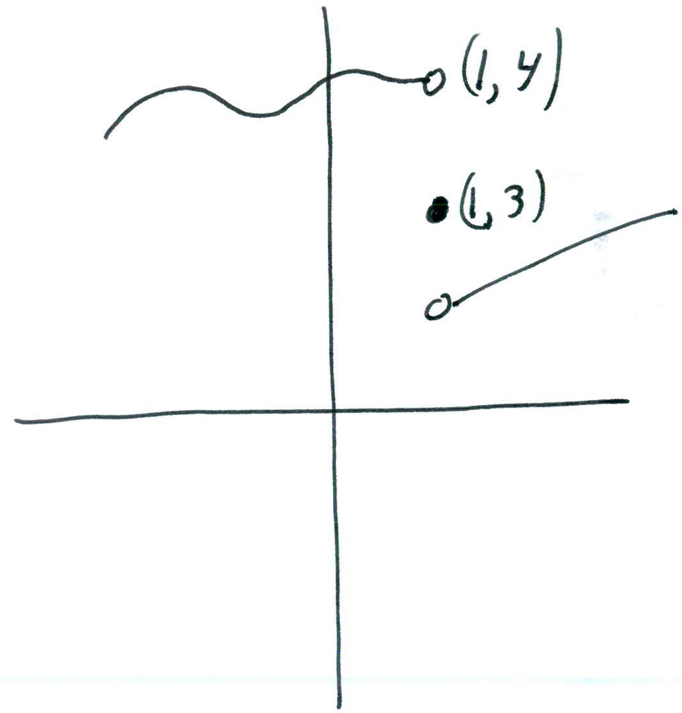
$$(x,y) = (1,2)$$

Plot those locations with open circles



Finally, determine what is being said about those locations, and add features to your graph that satisfy the descriptions.

- Point at $(x, y) = (1, 3)$
- Graph should appear to be heading for $(1, 4)$ from the left.
- Graph should appear to be heading for $(1, 2)$ from the right



Now Discuss Analytic Approach to Limits.
(function $f$ given by formula, not a graph).
See Reference ~~Page~~ 5 in Course Packet (page 9)

Example $f(x) = \underbrace{-7x^2 + 13x - 29}$
find $\lim\limits_{x \to 2} f(x)$.

observe $f$ is a polynomial

Solution

$$\lim_{x \to 2} f(x) \underset{\text{Theorem } 3}{=} f(2) = -7(2)^2 + 13(2) - 29 = -31$$

Example $f(x) = \sqrt{16 + x^2}$ Find $\lim\limits_{x \to 3} f(x)$.

Solution

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \sqrt{16 + x^2}$$

$$= \sqrt{\lim_{x \to 3} \underbrace{16 + x^2}_{\text{polynomial}}} \quad \text{by Theorem 2.8}$$

$$= \sqrt{16 + (3)^2} \quad \text{by Theorem 3}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

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So far, finding limit seems to be just like finding the y-value. But this is not usually the case. Now we'll do harder examples.

Examples involving the rational function

$$f(x) = \frac{x^2 + 3x - 10}{x - 2} = \frac{(x-2)(x+5)}{(x-2)}$$

(A) find $f(c)$ for $c = 0, 1, 2, 3$

Solution

$$f(0) = \frac{(0-2)(0+5)}{(0-2)} = \frac{\cancel{(-2)}(5)}{\cancel{(-2)}} = 5$$

$$f(1) = \frac{(1-2)(1+5)}{(1-2)} = \frac{\cancel{(-1)}(6)}{\cancel{(-1)}} = 6$$

$$f(2) = \frac{(2-2)(2+5)}{(2-2)} = \frac{(0)(7)}{(0)} =$$ Does not exist!

Cannot cancel $\frac{0}{0}$

$\frac{0}{0}$ is undefined!

$$f(3) = \frac{(3-2)(3+5)}{(3-2)} = \frac{(1)(8)}{(1)} = 8$$

Observe $x=2$ is <u>not</u> in the domain of $f$.
(So graph of $f$ does not have $y$-value at $x=2$).

(B) Find $\lim_{x \to c} f(x)$ for $c = 0, 1, 2$

Solution

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(x-2)(x+5)}{(x-2)}$$

$$= \lim_{x \to 0} x+5$$

Can cancel factors inside a limit (by the one x rule)

$$= 0 + 5$$

$$= 5$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}}$$

can cancel factors inside a limit (by the one x rule)

$$= \lim_{x \to 1} x+5$$

$$= 1 + 5$$

$$= 6$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}}$$

can cancel factors inside a limit. (by the one x rule)

$$= \lim_{x \to 2} x+5$$

$$= 2+5$$

$$= 7$$

Observe at $x=2$ there is a limit, even though there was no y-value.

Why does this make sense?

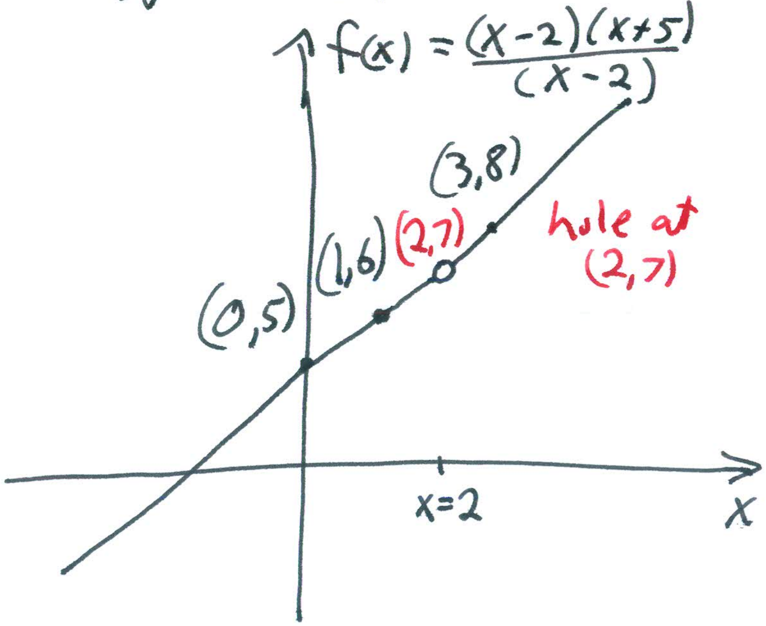
Graph $f(x) = \frac{(x-2)(x+5)}{(x-2)}$ and graph $g(x) = (x+5)$ side-by-side.

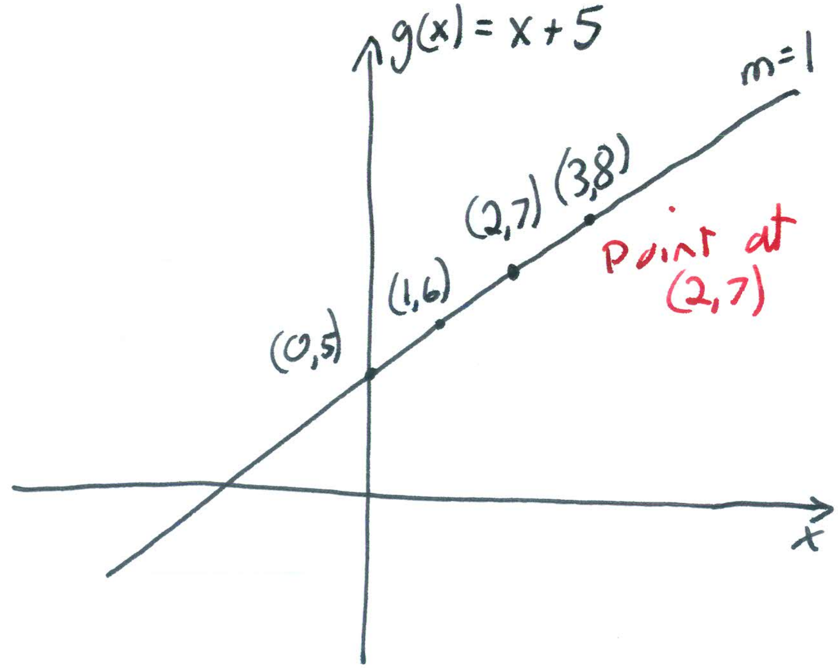
The graph of $g(x) = x + 5$ will be a

$y = x + 5$

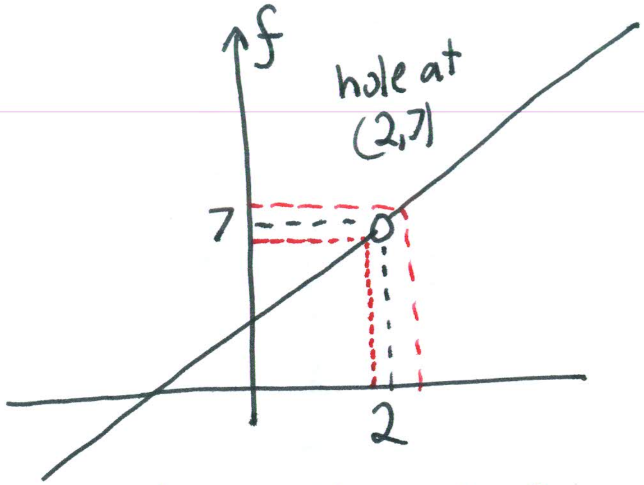
Straight line with slope $m = 1$ and y-intercept $(x, y) = (0, 5)$

The graph of $f$ will look just like the graph of $g$, but the graph of $f$ <u>won't</u> have a y-value at $x = 2$.



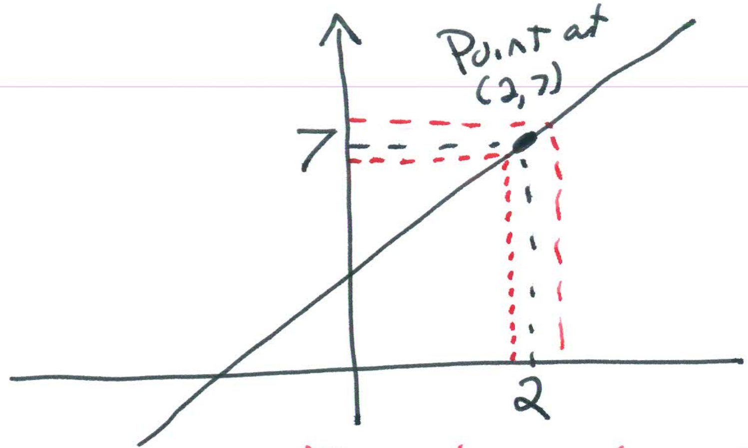



Consider $\lim_{x \to 2} f(x)$ and $\lim_{x \to 2} g(x)$ using graphs.




X values close to 2 but not equal to 2 cause y-values close to 7.

$$\lim_{x \to 2} f(x) = 7$$




X values close to 2 but not equal to 2 cause y-values close to 7

$$\lim_{x \to 2} g(x) = 7$$

So $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x)$

$$\lim_{x \to 2} \frac{(x-2)(x+5)}{(x-2)} = \lim_{x \to 2} (x+5)$$

This is the essence of the "one-x-Rule" on the list of rules in Reference 5.

But be <u>careful</u>! You <u>can</u> cancel <u>factors</u>, such as $\frac{x-2}{x-2}$, but you <u>cannot</u> <u>ever</u> cancel $\frac{0}{0}$ !!!

Example of a <u>bad</u> calculation

Find $\lim\limits_{x\to 2} \frac{(x-2)(x+5)}{x-2}$

Solution $\lim\limits_{x\to 2} \frac{(x-2)(x+5)}{(x-2)} = \frac{\cancel{(2-2)}(2+5)}{\cancel{(2-2)}} = 7$

not true    not true

This is an invalid computation.

# Examples involving "Difference Quotients"

**Example** $f(x) = x^2 - 6x + 5$

Find $\cancel{\lim_{x\to 2}}$ $\displaystyle\lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$

**Solution**

$$\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\left((4+h)^2 - 6(4+h) + 5\right) - \left((4)^2 - 6(4) + 5\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(4^2 + 8h + h^2 - 24 - 6h + 5\right) - \left(16 - 24 + 5\right)}{h}$$

$$= \lim_{h\to 0} \frac{8h + h^2 - 6h}{h}$$

$$= \lim_{h\to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$$ factored out an $h$.

can cancel factors inside limit

$$= \lim_{h \to 0} \underbrace{2+h}_{}$$

polynomial
can just substitute in $h=0$

$$= 2 + 0$$

$$= 2$$