Tuesday September 11, 2012

Continuing discussion of Rates of Change (Section 3-4)

Question where did this formula come from?

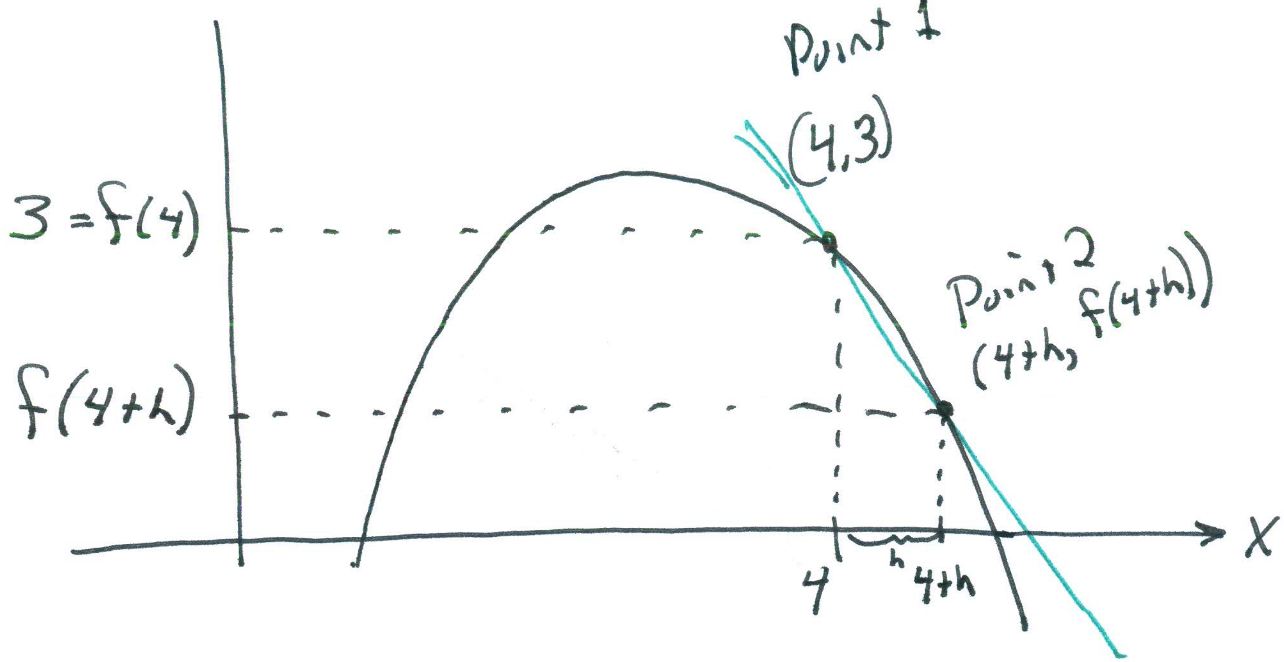
$$m = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

Why is that the formula for the slope of the tangent line?

Consider ~~the~~ two nearby points on the graph of f.

Point 1 is the point $(x, y) = (4, f(4)) = (4, 3)$

Point 2 is the point $(x, y) = (4+h, f(4+h))$



this line has slope

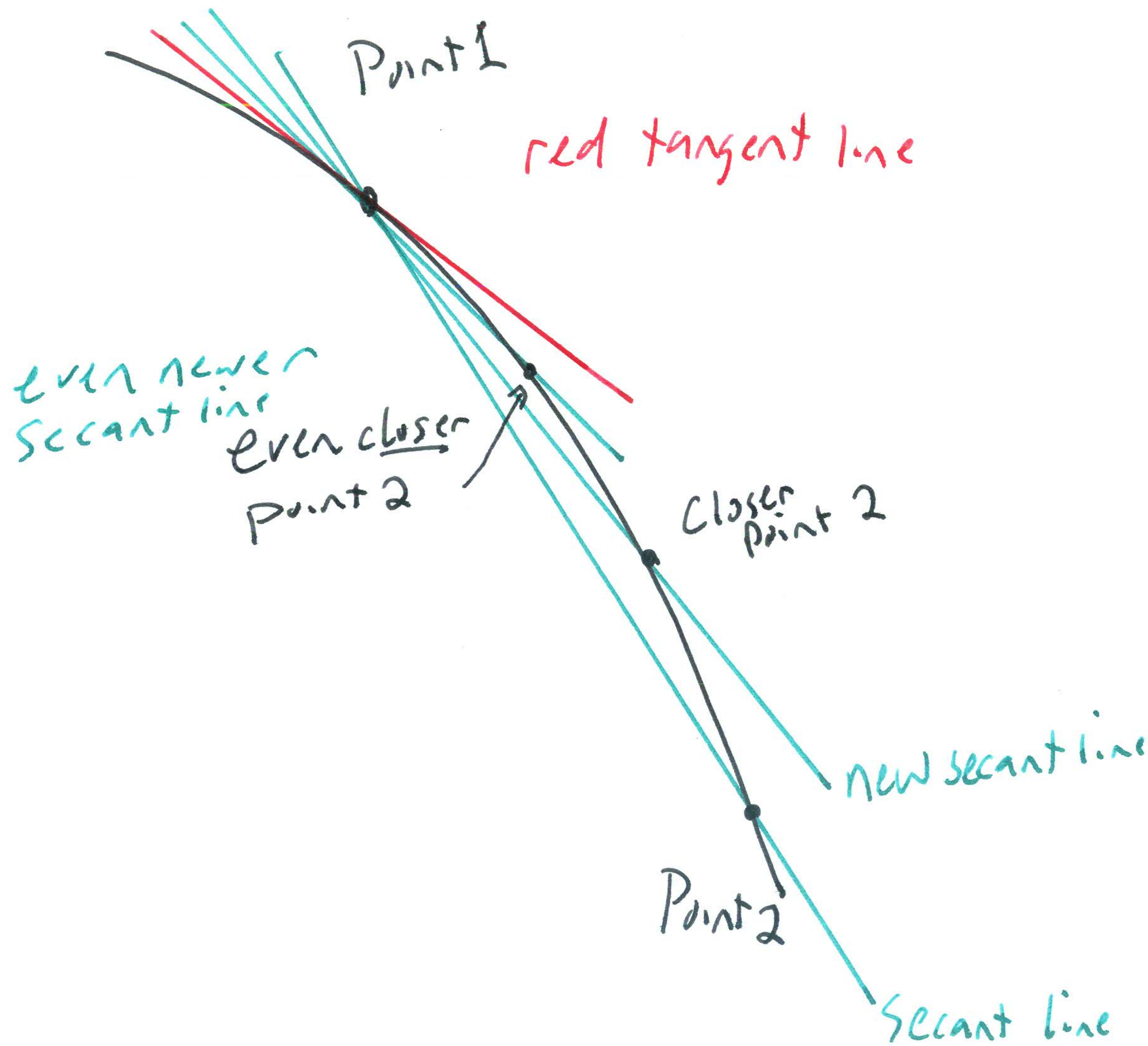
$$m = \frac{\Delta y}{\Delta x} = \frac{f(4+h) - f(4)}{(4+h) - 4}$$

these cancel

Secant line slope $m = \dfrac{f(4+h) - f(4)}{h} = -h - 2$

from yesterday

Now imagine pulling the Point 2 in closer & closer to Point 1



As we pull point 2 closer + closer to point 1, the number that is the secant line slope should get closer + closer to the number that is tangent line slope.

In mathematical symbols

this expression is a secant line slope

$$\text{tangent line slope } m = \underbrace{\lim_{h \to 0}}_{\text{bring 2nd point closer}} \underbrace{\frac{f(4+h) - f(4)}{h}}$$

Class Drill 4 Representations of Slopes

Class Drill 4: Representations of Slopes

In Section 3-4 of the textbook, you learned about average rate of change and instantaneous rate of change.

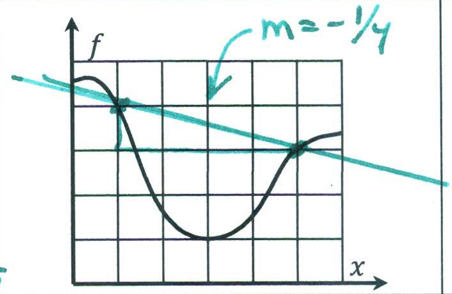
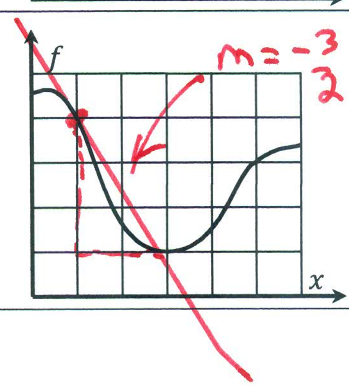
Definition of *Average Rate of Change*

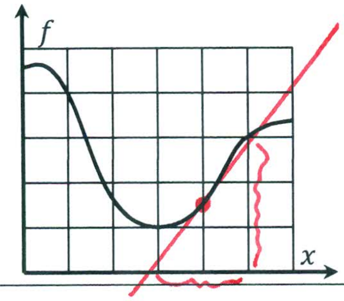
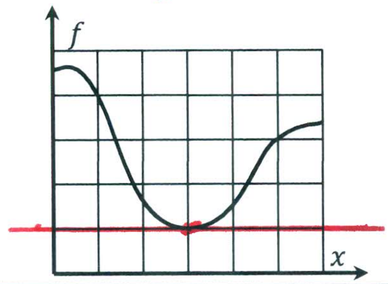
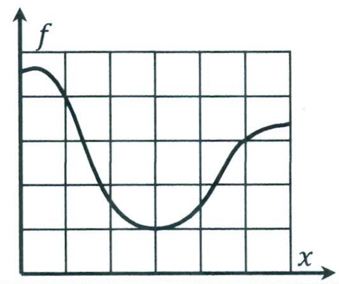
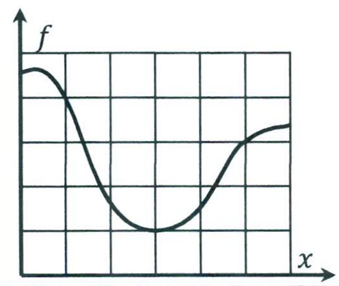
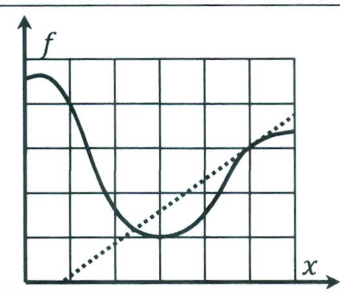
- **words:** the average rate of change of f as the input changes from a to b
- **usage:** f is a function that is continuous on the interval $[a, b]$.
- **meaning:** the number $m = \frac{f(b)-f(a)}{b-a}$
- **graphical interpretation:** The number m is the slope of the secant line that touches the graph of f at the points $(a, f(a))$ and $(b, f(b))$.
- **remark:** The average rate of change m is a number.

Definition of *Instantaneous Rate of Change*

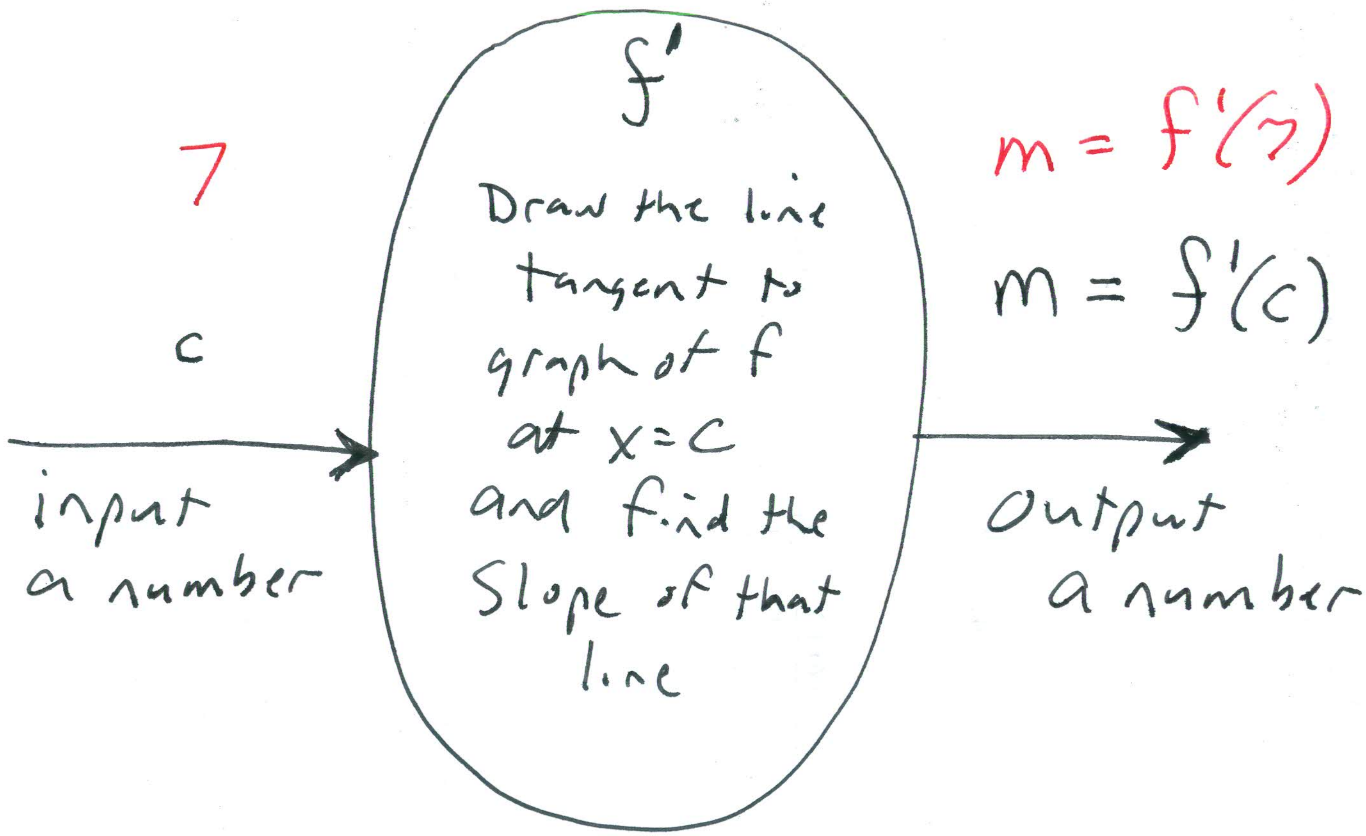
- **words:** the instantaneous rate of change of f at a
- **alternate words:** the derivative of f at a
- **symbol:** $f'(a)$
- **meaning:** the number $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
- **graphical interpretation:** The number m is the slope of the line tangent to the graph of f at the point $(x, y) = (a, f(a))$.
- **remark:** The instantaneous rate of change $f'(a)$ is a number.

Each expression in the left column represents a number m that can be interpreted as the slope of a line on the graph of f. In each example, draw the line on the graph of f, or write the missing expression based on the line shown in the graph, and then give the value of the number m represented by the expression.

Example	Expression representing m	Line whose slope is m	Value of m
(1)	the average rate of change of f as the input changes from 1 to 5 Slope of secant line that touches graph at $x=1$, $x=5$		$m = \frac{\Delta y}{\Delta x} = \frac{-1}{4}$
(2)	the derivative of f at $x = 1$ Slope of tangent line at $x=1$		$m = \frac{\Delta y}{\Delta x} \approx \frac{-3}{2}$

Example	Expression representing m	Line whose slope is m	Value of m
(3)	the instantaneous rate of change of f at $x = 4$ Slope of tangent line at $x=4$		$m = \frac{\Delta y}{\Delta x} \approx \frac{3}{2}$
(4)	$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ Slope of line tangent at $x=3$		$m = 0$ horizontal
(5)	$\frac{f(4) - f(2)}{4 - 2}$		$m =$
(6)	$f'(2)$		$m =$
(7)	$f'(5)$ The instantaneous rate of change at 5 $\frac{f(5) - f(2.5)}{5 - 2.5}$		$m = \frac{3}{4}$

Consider an abstract mathematical machine



This machine is called the **derivative of f**

Denoted by the symbol f'

Now Do Class Drill 5

Finding Derivatives Graphically Using a Ruler

(9)

Class Drill 5: Finding Derivatives Graphically Using a Ruler

The goal: Given the graph of f on the top axes on the next page, make a graph of f' on the bottom axes.

On the graph of f', the input will be x and the output will be $f'(x)$. Remember the graphical interpretation of $f'(x)$:

Definition of the *Derivative*

- **symbol:** $f'(a)$
- **graphical interpretation:** $f'(a)$ is the number that is the slope of the line tangent to the graph of f at the point where $x = a$.

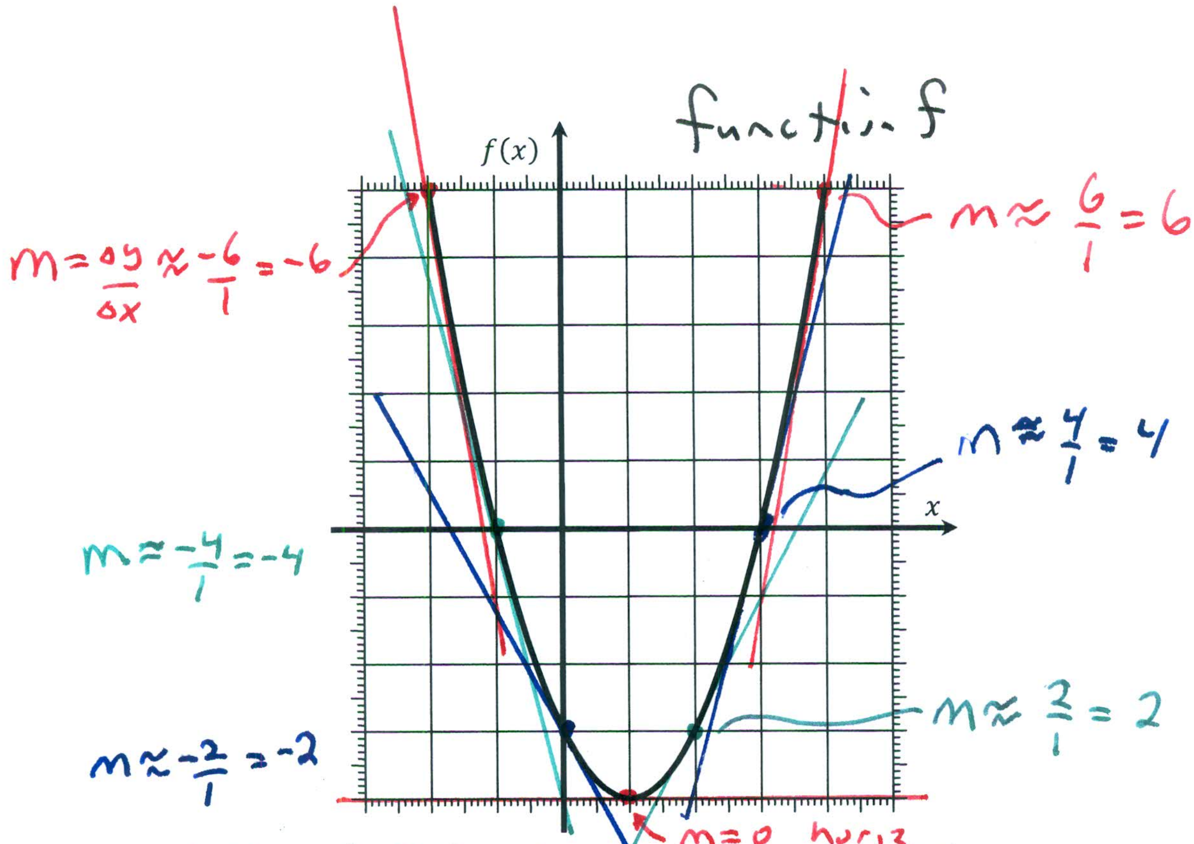
Part 1: Prepare the data for your graph of f' by filling out the following table.

x	what to do on the graph of f	$f'(x)$
-2	Draw the line tangent to the graph of f at the point where $x = -2$ and find its slope m. This slope m will be the value of $f'(-2)$.	-6
-1	Draw the line tangent to the graph of f at the point where $x = -1$ and find its slope m. This slope m will be the value of $f'(-1)$.	-4
0	Draw the line tangent to the graph of f at the point where $x = 0$ and find its slope m. This slope m will be the value of $f'(0)$.	-2
1	Draw the line tangent to the graph of f at the point where $x = 1$ and find its slope m. This slope m will be the value of $f'(1)$.	0
2	Draw the line tangent to the graph of f at the point where $x = 2$ and find its slope m. This slope m will be the value of $f'(2)$.	2
3	Draw the line tangent to the graph of f at the point where $x = 3$ and find its slope m. This slope m will be the value of $f'(3)$.	4
4	Draw the line tangent to the graph of f at the point where $x = -2$ and find its slope m. This slope m will be the value of $f'(4)$.	6

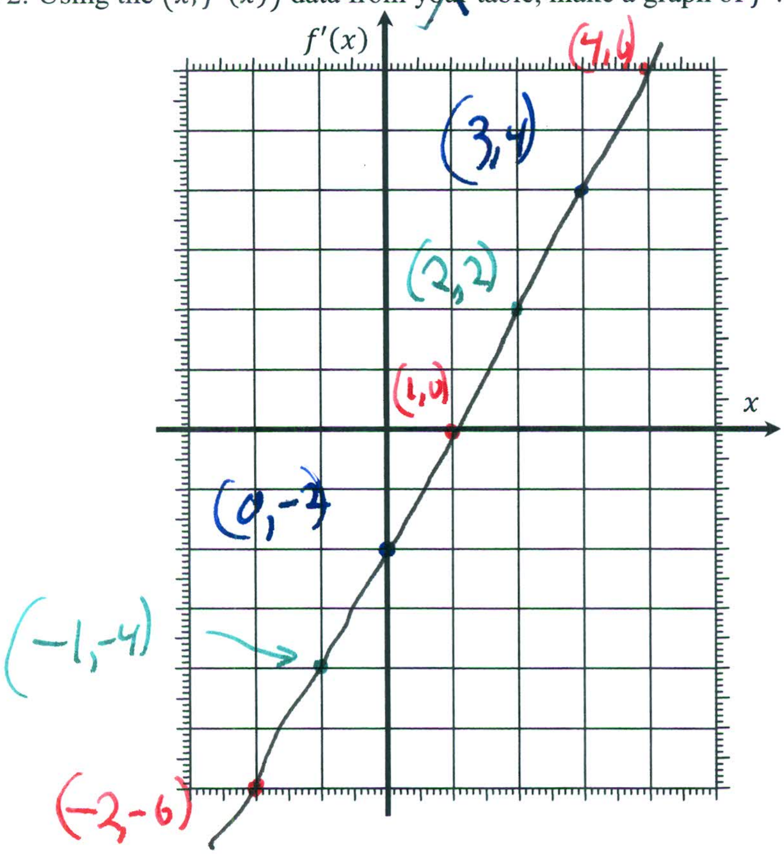
Part 2 is on the next page.

10

function f



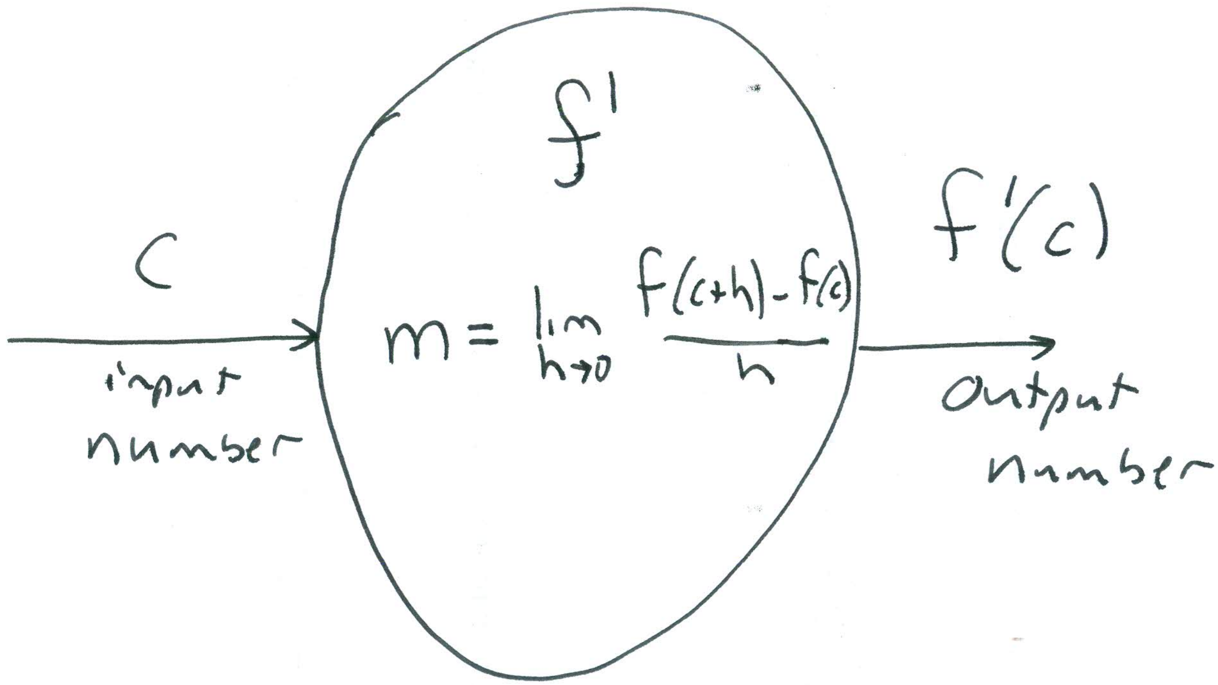
Part 2: Using the $(x, f'(x))$ data from your table, make a graph of f'.



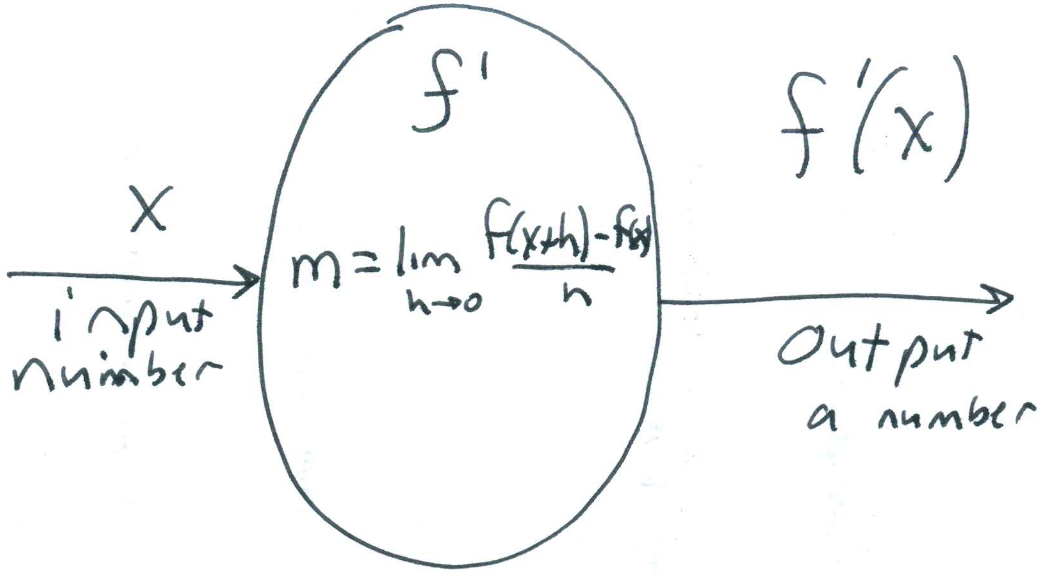
the Derivative of f

Called f'

Now consider the abstract mathematical version of this



Or use the number x for the input



For example let $f(x) = x^2 - 2x - 3$

find $f'(x)$ using the definition of the derivative.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\qquad\qquad) - (x^2 - 2x - 3)}{h}$$

Dirty work

$$f(x) = x^2 - 2x - 3$$

$$f(\ ) = (\ )^2 - 2(\ ) - 3$$ empty version

$$f(x+h) = (x+h)^2 - 2(x+h) - 3$$

$$= \underbrace{x^2 + 2xh + h^2 - 2x - 2h - 3}$$

this whole thing goes in the parentheses in $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 2x - 2h - 3) - (x^2 - 2x - 3)}{h}$$

these cancel

$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}$$

factor out an h in numerator

$$= \lim_{h \to 0} \frac{h(2x + h - 2)}{h}$$

these cancel

$$= \lim_{h \to 0} 2x + h - 2$$

Can substitute in $h = 0$

$$= 2x + 0 - 2$$

$$f'(x) = 2x - 2$$

Discuss this

We found that when $f(x) = x^2 - 2x - 3$,

the derivative is $f'(x) = 2x - 2$

What would these graphs look like?

$$f(x) = x^2 - 2x - 3 = (x+1)(x-3)$$

parabola facing up

y-intercept $(0, -3)$

x-intercepts at $x = -1$, $x = 3$

$$f'(x) = 2x - 2$$

Straight line with slope $m = 2$

y-intercept $b = -2$

This matches the graphs from Class Drill 5!!