Tuesday, September 18, 2012 (Day 10)

Yesterday: Learned our first three Differentiation Rules

Power Rule: $\frac{d}{dx} x^n = n \cdot x^{n-1}$

Sum + Constant Multiple Rules:

$$\frac{d}{dx}\left(a \cdot f(x) + b \cdot g(x)\right) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

a, b constants   f, g functions

Our last example was $f(x) = x^2 - 2x - 3$

We found $f'(x) = 2x - 2$. This agreed with what we found last week, using graphical techniques an the "definition of the derivative".

Continue this example $f(x) = x^2 - 2x - 3$

(A) Yesterday's example: $f'(x) = 2x - 2$

(B) Find $f'(3)$ and $f'(2.5)$ and $f'(0)$
and interpret these results graphically.

Solution

$f'(3)$ means to substitute $x=3$ into
the function $f'(x) = 2x - 2$

- $f'(3) = 2(3) - 2 = 6 - 2 = 4 = f'(3)$

$f'(2.5)$ means to substitute $x = 2.5$ into f'

- $f'(2.5) = 2(2.5) - 2 = 5 - 2 = 3 = f'(2.5)$
- $f'(0) = 2(0) - 2 = 0 - 2 = -2 = f'(0)$

Graphical interpretation:

The line tangent to graph of f at $x=3$ has slope $m=f'(3)=4$

" " " " " " " at $x=2.5$ " " $m=f'(2.5)=3$

" " " " " " " at $x=0$ " " $m=f'(0)=-2$

(C) Find all x-values where the tangent line is horizontal.

Solution

Reword the question

Find all x-values where the tangent line has slope $m=0$.

that is

Find all x-values where $f'(x)=0$,

$$2x-2=0 \quad \text{Solve for } x.$$

$$2x=2$$

$$x=1$$

This agrees with graph.

Harder Derivative Problems

Example 1 Let $f(x) = \dfrac{23x^6}{3} + \dfrac{19x}{5} + \dfrac{17}{7x} + \dfrac{13}{11x^6}$

Find $f'(x)$

Solution Start by rewriting $f(x)$ as constants & power functions. Fundamental Skill from Algebra

$$f(x) = \left(\frac{23}{3}\right)x^6 + \left(\frac{19}{5}\right)x + \left(\frac{17}{7}\right)\left(\frac{1}{x}\right) + \left(\frac{13}{11}\right)\left(\frac{1}{x^6}\right)$$

$$= \left(\frac{23}{3}\right)x^6 + \left(\frac{19}{5}\right)x + \left(\frac{17}{7}\right)x^{-1} + \left(\frac{13}{11}\right)x^{-6}$$

Now find $f'(x)$

$$f'(x) = \frac{d}{dx}\left(\left(\frac{23}{3}\right)x^6 + \left(\frac{19}{5}\right)x + \left(\frac{17}{7}\right)x^{-1} + \left(\frac{13}{11}\right)x^{-6}\right)$$

$$= \left(\frac{23}{3}\right)\left(\frac{d}{dx}x^6\right) + \left(\frac{19}{5}\right)\left(\frac{d}{dx}x\right) + \left(\frac{17}{7}\right)\left(\frac{d}{dx}x^{-1}\right) + \left(\frac{13}{11}\right)\left(\frac{d}{dx}x^{-6}\right)$$

Used sum & Constant Multiple Rule

$$= \left(\frac{23}{3}\right)\left(6x^{5}\right) + \left(\frac{19}{5}\right)\left(1\right) + \left(\frac{17}{7}\right)\left(-1x^{-1-1}\right) + \left(\frac{13}{11}\right)\left(-6x^{-6-1}\right)$$ Used Power Rule

$$= \frac{(23)6x^{5}}{3} + \frac{19}{5} - \frac{17}{7}x^{-2} - \frac{13(6)}{11}x^{-7}$$ clean up

$$f'(x) = (23)(2)x^{5} + \frac{19}{5} - \frac{17}{7x^{2}} - \frac{13(6)}{11x^{7}}$$ rewrite with positive exponents

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Example 2 $\quad f(x) = \dfrac{2\sqrt[5]{x}}{7} - \dfrac{3}{11x^{3/5}}$ find $f'(x)$.

Solution Rewrite f as constants & power functions.

$$f(x) = \left(\frac{2}{7}\right)\sqrt[5]{x} - \left(\frac{3}{11}\right)\frac{1}{x^{2/5}}$$

$$= \left(\frac{2}{7}\right)x^{\frac{1}{5}} - \frac{3}{11}\cdot x^{-2/5}$$

Now find $f'(x)$

$$f'(x) = \left(\frac{2}{7}\right)\left(\frac{d}{dx}x^{1/5}\right) - \left(\frac{3}{11}\right)\left(\frac{d}{dx}x^{-2/5}\right)$$

$\leftarrow n = \frac{1}{5}$ $\quad$ $\leftarrow n = -\frac{2}{5}$

already used the sum & constant multiple rule.

$$= \left(\frac{2}{7}\right)\left(\frac{1}{5}\cdot x^{\frac{1}{5}-1}\right) - \left(\frac{3}{11}\right)\left(\left(-\frac{2}{5}\right)x^{-\frac{2}{5}-1}\right)$$

Power Rule

$$= \left(\frac{2}{35}\right)x^{-\frac{4}{5}} + \left(\frac{6}{55}\right)x^{-\frac{7}{5}}$$

$$= \left(\frac{2}{35}\right)\frac{1}{(x^{4/5})} + \left(\frac{6}{55}\right)\frac{1}{x^{7/5}}$$

rewrote with positive exponents

$$f'(x) = \frac{2}{35x^{4/5}} + \frac{6}{55x^{7/5}}$$

Example 3 Trick Problem!!

$$f(x) = \frac{2x^5 - 4x^3 + 2x}{x^3}$$ Find $f'(x)$.

Solution Rewrite $f(x)$

$$f(x) = \frac{2x^5}{x^3} - \frac{4x^3}{x^3} + \frac{2x}{x^3}$$

$$= 2x^2 - 4 + \frac{2}{x^2}$$

$$= 2x^2 - 4(1) + 2x^{-2}$$ constants & power functions.

From here, the derivative is easy

$f'(x) = \ldots$ just like previous two problems.

$$f'(x) = \frac{d}{dx}\left(2x^2 - 4(1) + 2x^{-2}\right)$$

$$= 2\left(\frac{d}{dx}x^2\right) - 4\left(\frac{d}{dx}1\right) + 2\left(\frac{d}{dx}x^{-2}\right)$$

$$= 2(2x) - 4(0) + 2(-2x^{-2-1})$$

$$= 4x + 0 - 4x^{-3}$$

$$= 4x - \frac{4}{x^3}$$

More Tangent Line Problems

Example $f(x) = -x^3 + 9x^2 - 23x + 15$

(A) find the slope of the line that is tangent to graph of f at $x=2$.

Solution We need to find $m = f'(2)$

Strategy:
- find $f'(x)$
- Substitute $x=2$ into $f'(x)$

$$f'(x) = \frac{d}{dx}\left(-x^3 + 9x^2 - 23x + 15\right)$$

$$= (-3x^2) + 9(2x) - 23(1) + 15(0)$$

$$= -3x^2 + 18x - 23$$

Now substitute $x=2$ into $f'(x)$

$$f'(2) = -3\cdot 2^2 + 18(2) - 23$$

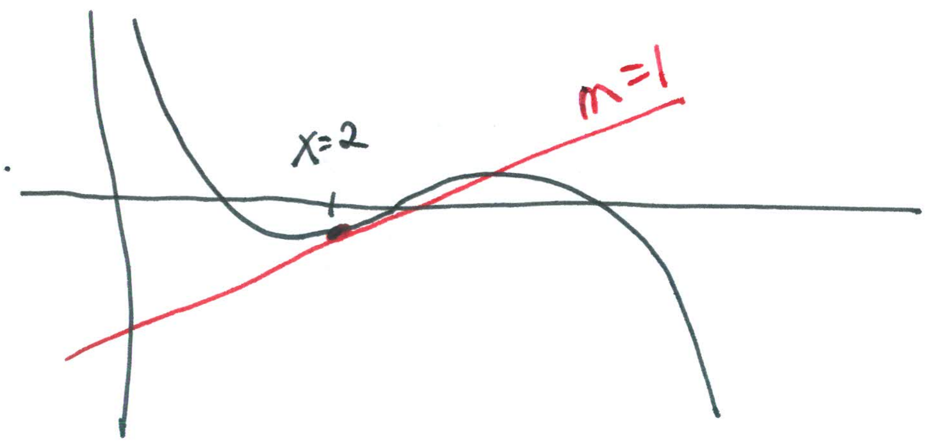
$$= -3\cdot 4 + 36 - 23$$

$$= -12 + 36 - 23$$

$$= 24 - 23$$

$$= 1$$

This agrees with graph from computer



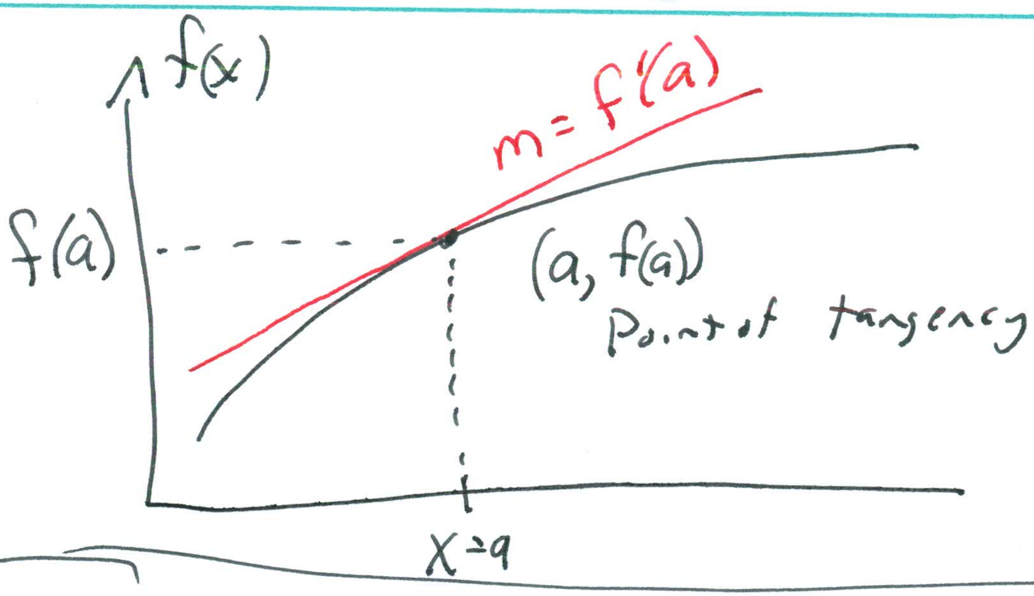
(B) Find the equation for the line tangent at $x=2$

~~Re~~ Background: Tangent Line Theory

- Recall, from previous courses, the "Point Slope Form" of the equation for the line that has known point (a, b) and known slope m

$$(y-b) = m(x-a)$$

- What do we know about the line that is tangent to graph of f at $x=a$.
  - It must touch graph at point $(a, f(a))$
  - It has slope $m = f'(a)$



So, the equation for the tangent line is

$$(y - f(a)) = f'(a)(x - a)$$

$f(a)$: known y-value

$f'(a)$: known slope

a: known x-value

Now Return to our problem

We need to find the equation for the line tangent to graph of $f(x) = -x^3 + 9x^2 - 23x + 15$ at $x = 2$

Solution

We need to build this equation

$$(y - f(a)) = f'(a)(x - a)$$

Part 1 Find all the parts

$a = 2$ (this is the x-coordinate of the point of tangency)

$$f(a) = f(2) = -2^3 + 9(2)^2 - 23(2) + 15$$

$$= -8 + 9 \cdot 4 - 46 + 15$$

$$= -8 + \underbrace{36 - 46} + 15$$

$$= -8 - 10 + 15$$

$$= -3$$ this is the y-coordinate of point of tangency

$$f'(a) = f'(2) = 1$$

↑ result from awhile ago, in part (A)

Part 2 Now Assemble the Equation

$$(y - (-3)) = 1\,(x - 2)$$

This is an equation for the tangent line.

Convert to slope intercept form

$$y + 3 = x - 2$$

$$y = x - 5$$

Observe! this line has slope $m = 1$

and y-intercept $(0, -5)$

This agrees with the graph!