Day 33 is Tuesday, Nov 13, 2012

Start with a couple of examples using Section 6-2 techniques.

Example similar to 6-2 #75 (Biology)

A yeast culture weighs 1gm initially and is growing at a rate of

$$W'(t) = .3e^{(.1t)} \text{ gm/hour}.$$

(A) Find the weight function $W(t)$

(B) Find the weight of the culture at time $t = 24$ hours.

Solution

(A) We must find the particular antiderivative of $W'(t) = .3e^{(.1t)}$ that satisfies $W(0) = 1$.

Start by getting the <u>general</u> antiderivative.

$$W(t) = \int W'(t)\,dt$$

$$= \int .3 e^{(.1t)}\,dt$$

$$= .3 \int e^{(.1t)}\,dt$$

integrand: $e^{(.1t)}$ <u>nested</u>

inner$(t) = .1t$

outer$'(\ ) = e^{(\ )}$

therefore outer$(\ ) = e^{(\ )}$

<u>First try:</u>

$W(t) = .3 \cdot \text{outer}(\text{inner}(t)) + C$

$= .3 \cdot e^{(.1t)} + C$

Check:

$$W'(t) = \frac{d}{dt} W(t) = \frac{d}{dt}\left(.3e^{(.1t)} + C\right) = .3\frac{d}{dt} e^{(.1t)}$$

Chain rule work

$inner(t) = .1t$

$inner'(t) = .1$

$outer(\ ) = e^{(\ )}$

$outer'(\ ) = e^{(\ )}$

$W'(t) = (.3)e^{(.1t)} \cdot (.1)$

This is not what we wanted! We wanted $W'(t) = .3e^{.1t}$

We are off by a factor of .1.

So divide old $W(t)$ by .1 to get a new $W(t)$ and try again

2nd try $W(t) = \dfrac{.3e^{.1t}}{(.1)} + C = 3e^{.1t} + C$

Check

$W'(t) = \dfrac{d}{dt}W(t) = \dfrac{d}{dt}\left(3e^{(.1t)} + C\right) = 3\dfrac{d}{dt}e^{(.1t)}$

$= 3e^{(.1t)} \cdot (.1) = .3e^{.1t}$ ✓

Conclude that the general antiderivative is

$$W(t) = 3e^{.1t} + C$$

But we need the particular antiderivative that satisfies $W(0) = 1$.

Turn this around

$$\begin{aligned} 1 &= W(0) \\ &= 3e^{(.1)(0)} + C \\ &= 3e^{0} + C \\ &= 3(1) + C \\ 1 &= 3 + c \end{aligned}$$

therefore $c = -2$

So the particular antiderivative is

$$W(t) = 3e^{.1t} - 2$$

(B) Weight at 24 hours is

$$W(24) = 3e^{(.1)(24)} - 2$$

$$= 3e^{2.4} - 2$$

$$\approx 31 \text{ gm}$$

Compare part (A) solution to book's method.

$$W'(t) = .3e^{(.1t)}$$

Goal Find $W(t) = \int W'(t)\,dt$

$$= \int .3 e^{(.1t)} dt \Longrightarrow$$

Substitution box

let $u = .1t$

We need to find new version of $.3dt$

Take $\frac{d}{dt}$ of both sides

$$\frac{du}{dt} = \frac{d}{dt}.1t$$

$$\frac{du}{dt} = .1$$

Multiply both sides of equation
by dt

$$du = .1\,dt$$

But we need $.3dt$ So
Multiply both sides of equation
by 3

$$3\,du = .3\,dt$$

Substitute
this stuff in

$$= \int e^u \, 3\,du$$

$$= 3\int e^u\,du$$

$$= 3e^u + C$$

Substitute back
$u = .1t$

$$W(t) = 3e^{.1t} + C$$

This agrees with our previous solution.

Section 6.4 The Definite Integral

Let $f(x)$ be a positive function. (So the graph of f floats above the x-axis.

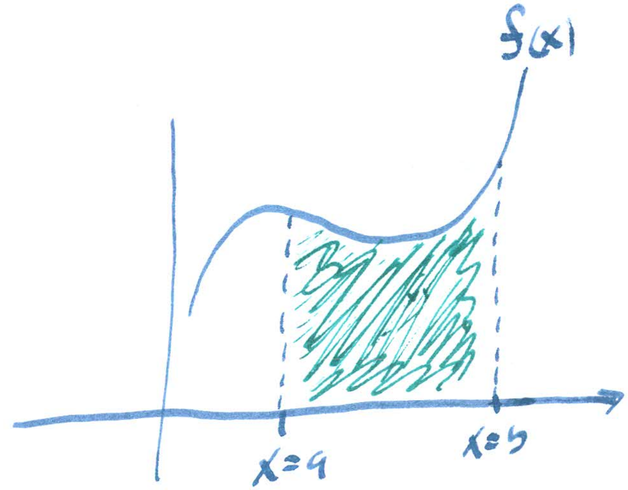
Define The Definite Integral

Symbol: $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$

Spoken: The definite integral, from $x=a$ to $x=b$, of $f(x)$.

Meaning: The area of the region bordered by

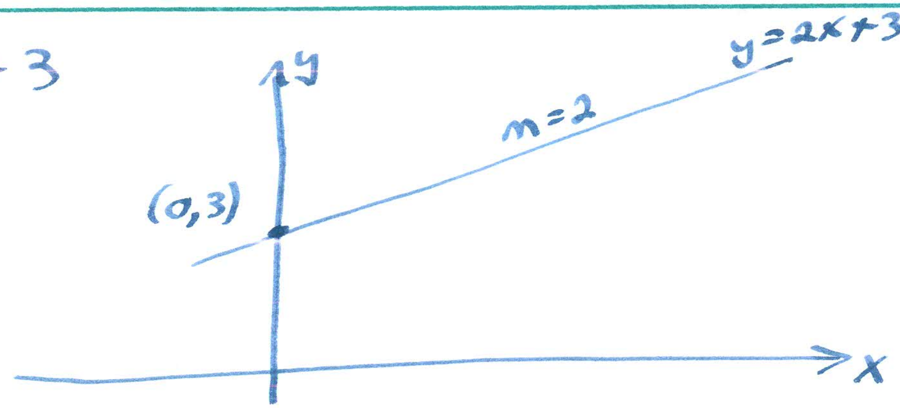
- the x-axis
- the graph of f
- the vertical line $x=a$
- the vertical line $x=b$



Observe: The symbol $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$

stands for a number.

Example Let $f(x) = 2x + 3$

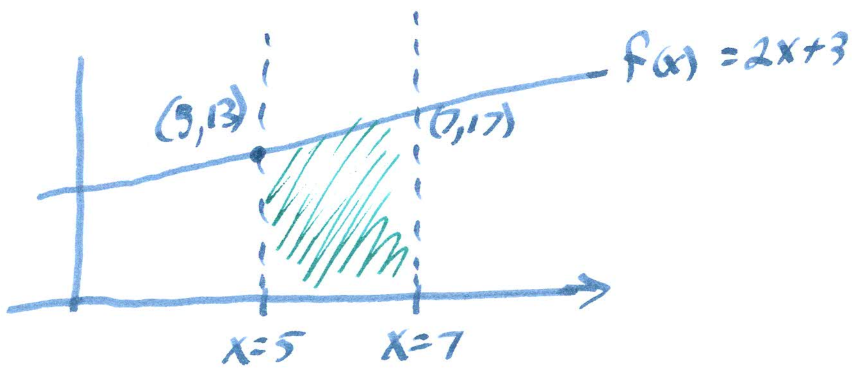


Find $\displaystyle\int_{x=5}^{7} f(x)\,dx$

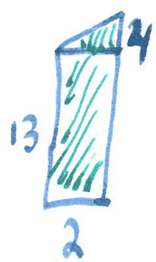
Solution

This symbol stands for a number. Call it I.

$$I = \int_{x=5}^{7} f(x)\,dx =$$



$=$



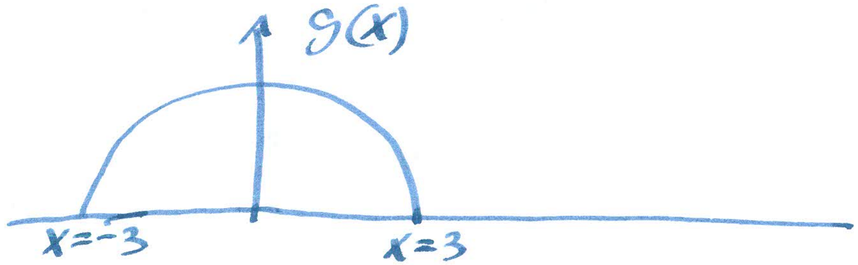
= rectangle + triangle

$= 2 \cdot 13 + \frac{1}{2}(2)(4)$

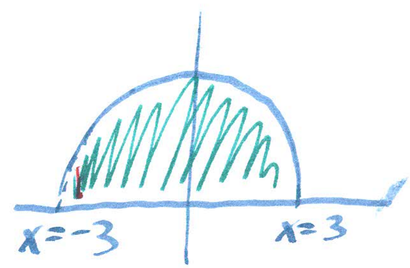
$= 26 + 4$

$I = 30$

Example let $g(x)$ be the function whose graph is the semicircle shown



Find: $\displaystyle\int_{x=-3}^{x=3} g(x)\,dx.$

Solution $I = \displaystyle\int_{x=-3}^{x=3} g(x)\,dx =$ 

$= \frac{1}{2}$ area of circle

$= \frac{1}{2} \cdot \pi r^2 = \frac{1}{2}\pi 9 = \frac{9\pi}{2}$

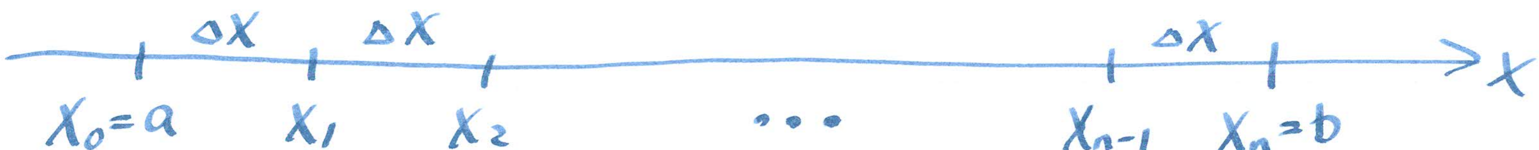
Question: What if graph of f is not a simple shape?

How do we find the area $I = \int_{x=a}^{x=b} f(x)\,dx$?

Introduce Riemann Sums

Divide the interval $a \leq x \leq b$ into n equal subintervals. For example, if $n=6$, it would look like this.

In general, for n subintervals



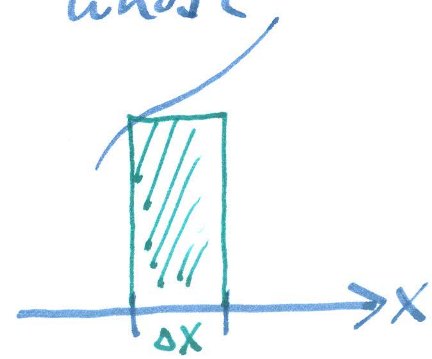
Width of each subinterval is

$$\Delta x = \frac{\text{width of whole interval}}{\text{number of subintervals}}$$

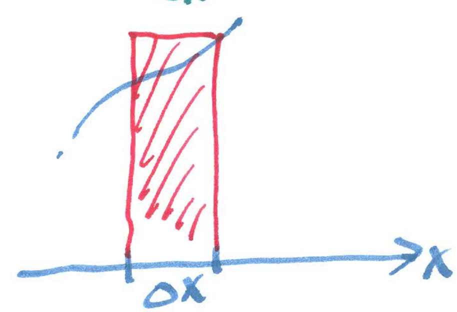
$$\boxed{\Delta x = \frac{b-a}{n}}$$

Park a rectangle on top of each subinterval.

"Left Rectangles" are rectangles whose left edge touches graph of f.



"Right Rectangles" are rectangles whose right edges touch graph of f



Define Left Sum and Right Sum
(examples of "Riemann Sums")

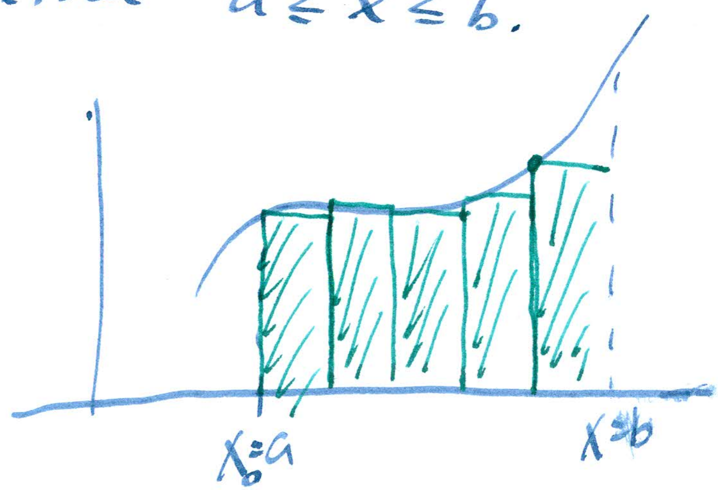
Given a definite integral $I = \int_{x=a}^{x=b} f(x)\,dx$.

Define the <u>Left Sum with n-subintervals</u>

to be the area~~s~~ of n left rectangles put on the interval $a \le x \le b$.

Symbol for Left Sum

↓



L_n = area of 1st rectangle + area of 2nd rectangle + ... + area of n^{th} rectangle

$= \Delta X f(X_0) + \Delta X \cdot f(X_1) + \cdots\cdots + \Delta X f(X_{n-1})$

this can be factored

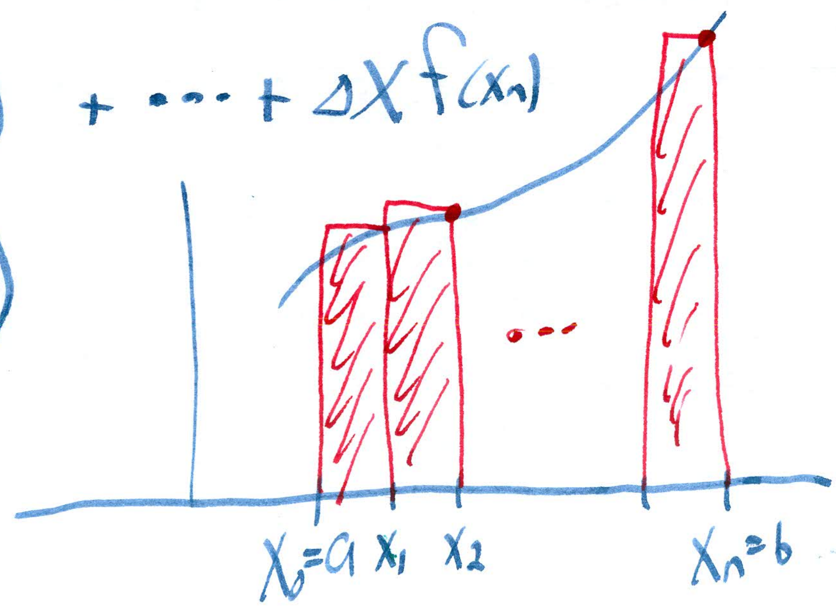
$$L_n = \Delta x\left(f(x_0) + f(x_1) + \cdots\cdots + f(x_{n-1})\right)$$

Analogously, we define the "Right Sum"

R_n = the sum of the areas of n right rectangles

$$= \Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_n)$$

$$= \Delta x\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)$$

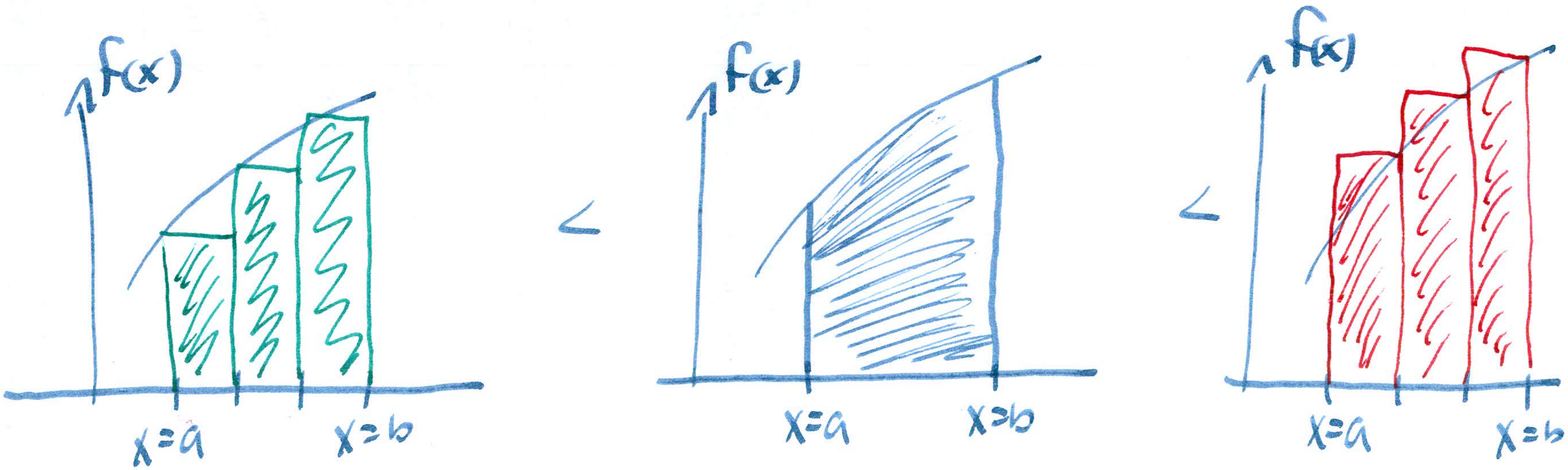


Key observation: If f is an increasing function,

then $\quad L_n < I < R_n$

$$L_n < \int_{x=a}^{x=b} f(x)\,dx < R_n$$

<u>Example with $n=3$</u>



$$L_3 < \int_a^b f(x)\,dx < R_3$$

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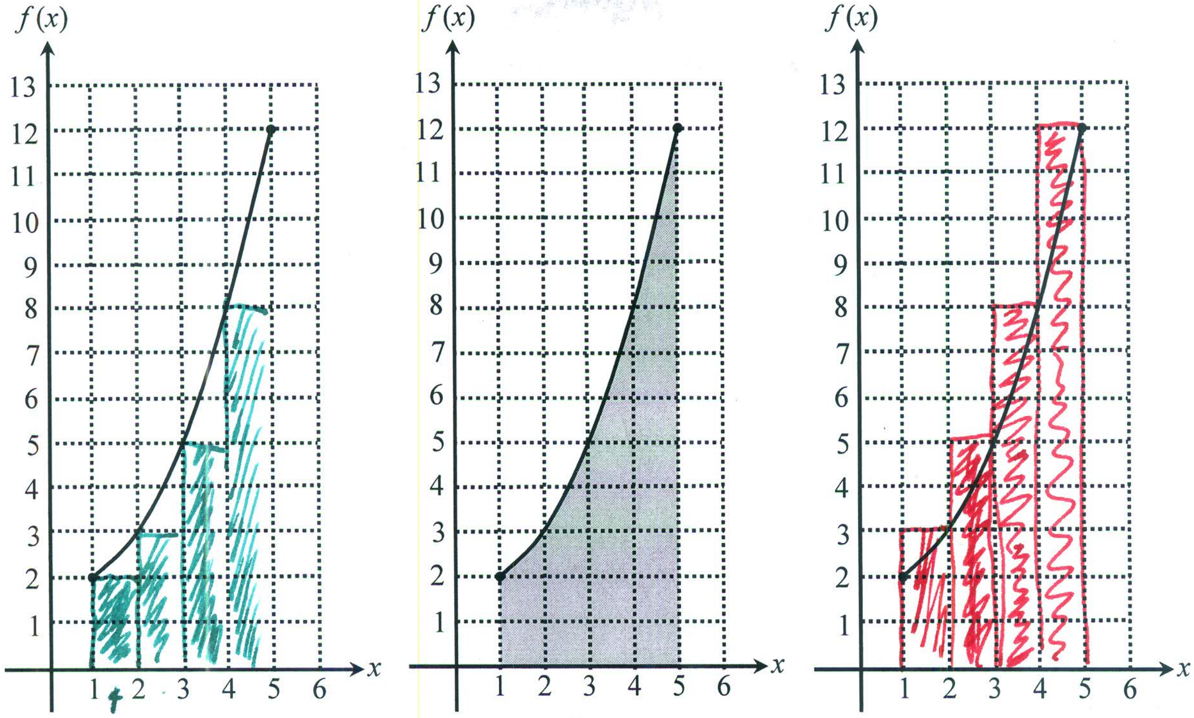
Class Drill 13: Estimating the Area Under a Graph Using Riemann Sums.

The goal is to estimate the shaded area in the middle figure. You will do this by finding the values of the Riemann sums L_4 and R_4. This will give you lower and upper bounds for the shaded area.



L_4 $\quad$ $A = \int_1^5 f(x)dx$ $\quad$ R_4

(A) Draw in the rectangles for the left sum L_4.

(B) Find the value of L_4. $L_4 = 2 + 3 + 5 + 8 = 18$

(C) Draw in the rectangles for the right sum R_4.

(D) Find the value of R_4. $R_4 = 3 + 5 + 8 + 12 = 28$

(E) In the expression $L_4 \leq \int_1^5 f(x)dx \leq R_4$, replace the symbols L_4 and R_4 with the values from questions (B)and (D).

$$\underline{18} \leq \int_1^5 f(x)dx \leq \underline{28}$$