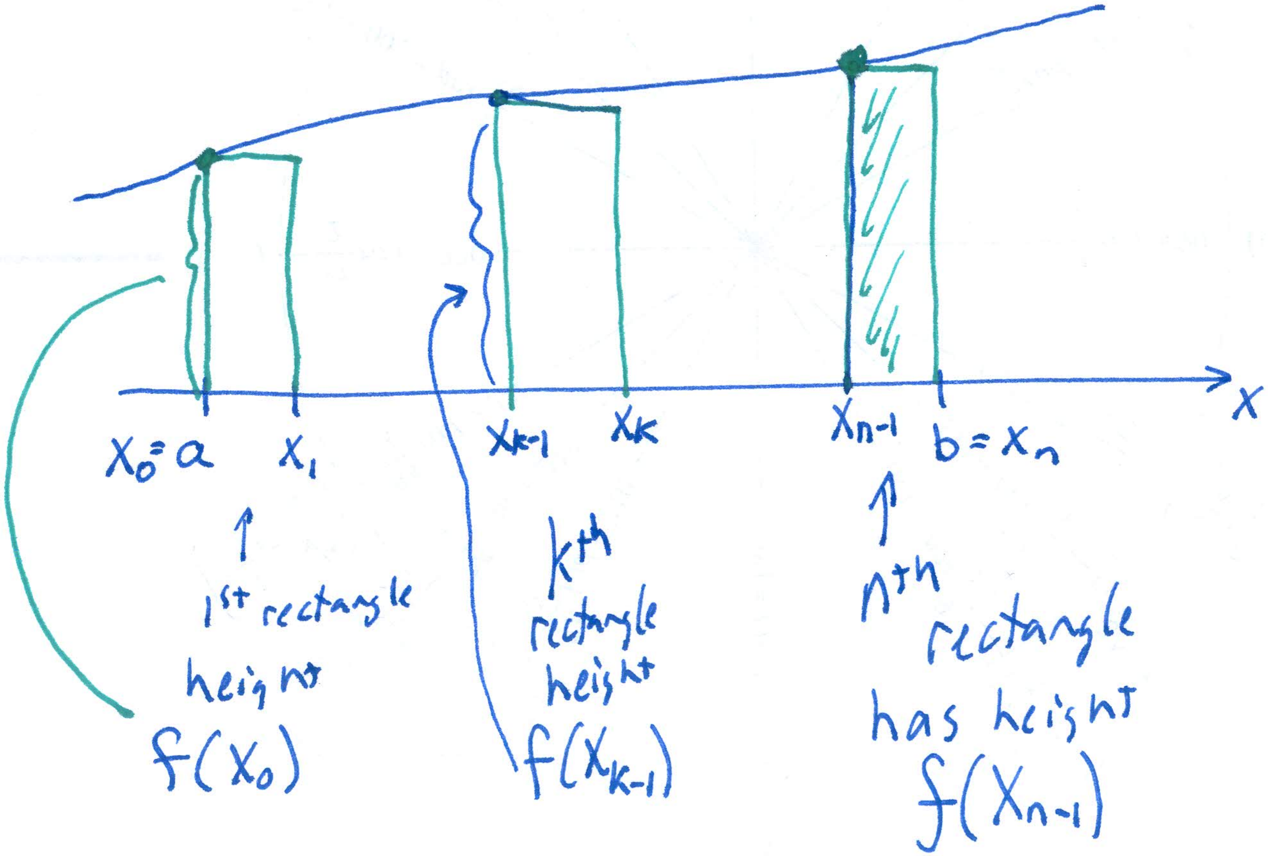
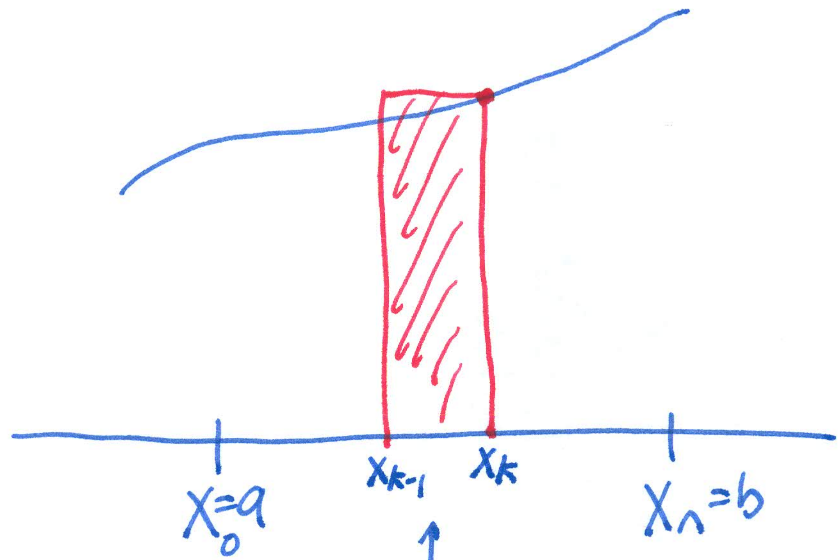
Day 34 is Thursday, Nov 15, 2012

Observe that for a Left Sum, the k^{th} rectangle has height $f(x_{k-1})$



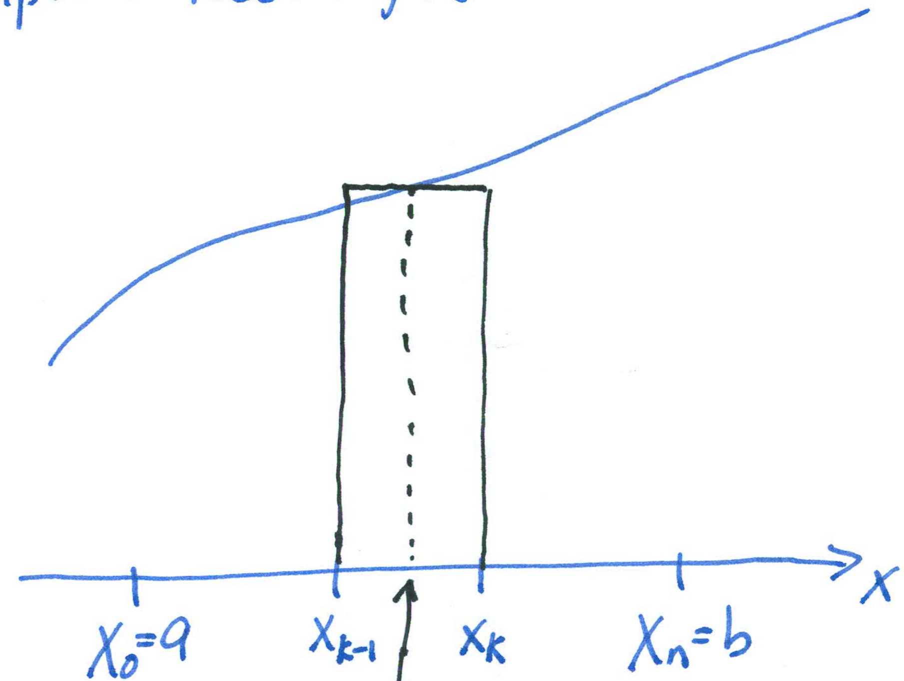
For Right Sum R_n, the the k^{th} rectangle has height $f(x_k)$



k^{th} rectangle has height $f(x_k)$

We can also have rectangles that touch the graph at some <u>other</u> x-value in the interval.

<u>Example</u> Midpoint Rectangle



the <u>midpoint</u> of the interval has x-coordinate $\dfrac{X_{k-1} + X_k}{2}$, the average of the x-values of endpoints

So the height of the rectangle is

$f(c_k)$ where $c_k = \dfrac{X_{k-1} + X_k}{2}$

Define Midpoint Sum M_n to be the sum of the areas of the midpoint rectangles.

So far, we have seen Left Sums, Right Sums and Midpoint Sums. L_n, R_n, M_n.

These are all examples of "Riemann Sums"

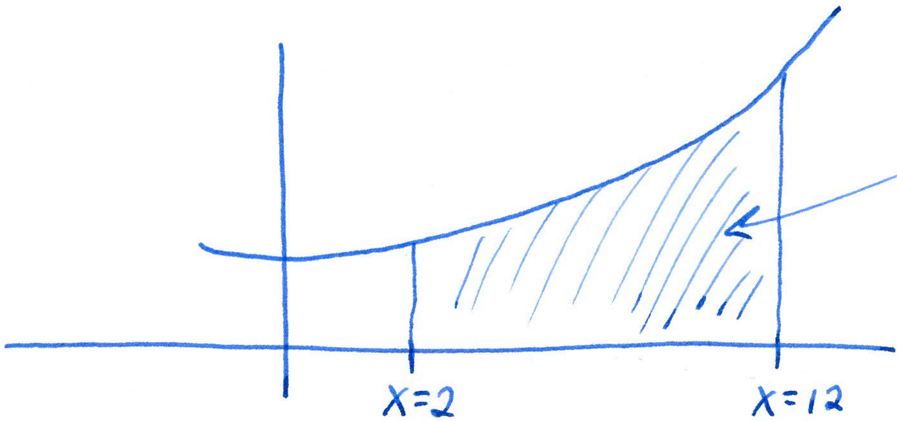
General Symbol for Riemann Sum: S_n.

(could be L_n, R_n, M_n, or some other kind)

Example of Riemann Sum where function f(x) is given by a formula.

Let $f(x) = 5 + \frac{x^2}{10}$

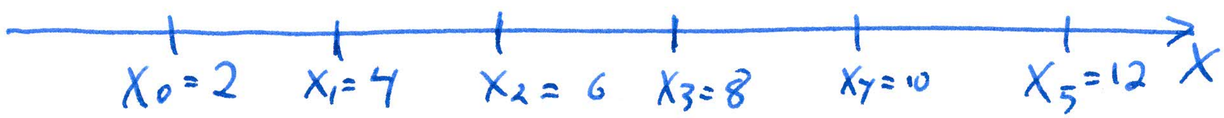
We are interested in this area: $A = \int_{x=2}^{x=12} f(x)\,dx$



Compute L_5

Solution Subdivide the interval

$\Delta x = \frac{b-a}{n} = \frac{12-2}{5} = \frac{10}{5} = 2$



$$L_5 = \underbrace{\Delta x}_{\text{width}} \cdot \underbrace{f(x_0)}_{\text{height}} + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4)$$

$$= \Delta x \left(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right)$$

We know $\Delta x = 2$

Find the five y-values

$$f(x_0) = f(2) = 5 + \frac{2^2}{10} = 5 + \frac{4}{10} = 5 + .4 = 5.4$$

$$f(x_1) = f(4) = 5 + \frac{4^2}{10} = 5 + \frac{16}{10} = 5 + 1.6 = 6.6$$

$$f(x_2) = f(6) = 5 + \frac{6^2}{10} = 5 + \frac{36}{10} = 5 + 3.6 = 8.6$$

$$f(x_3) = f(8) = 5 + \frac{8^2}{10} = \cdots = 11.4$$

$$f(x_4) = f(10) = 5 + \frac{10^2}{10} = 15$$

So $L_5 = 2(5.4 + 6.6 + 8.6 + 11.4 + 15) = 94$

Similarly, find R_5

Solution $R_5 = \Delta x\left(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right)$

Observe, we already know Δx, $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$

we only need $f(x_5)$

$$f(x_5) = f(12) = 5 + \frac{12^2}{10} = 5 + \frac{144}{10} = 5 + 14.4 = 19.4$$

So the sum is

$$R_5 = 2(6.6 + 8.6 + 11.4 + 15 + 19.4) = 122$$

Observe that

$$L_5 < \text{Actual Area } A < R_5$$

$$94 < \int_{x=2}^{x=12} f(x)\,dx < 122$$

Try Using More Rectangles!

$$L_{20} < A < R_{20}$$

$$103.4 < A < 111$$

Use Even More!

$$L_{100} < A < R_{100}$$

$$106.6 < A < 108$$

It seems that as $n \to \infty$, the values of L_n and R_n are getting closer + closer together, closer + closer to some common number.

So we will Define the area $A = \int_{x=a}^{x=b} f(x)\,dx$

to be the limit: $\lim\limits_{n\to\infty} L_n$ or $\lim\limits_{n\to\infty} R_n$

The general expression is this:

$$A = \int_{x=a}^{x=b} f(x)\,dx \underset{\text{definition}}{=} \lim_{n\to\infty} S_n \quad \text{where } S_n$$

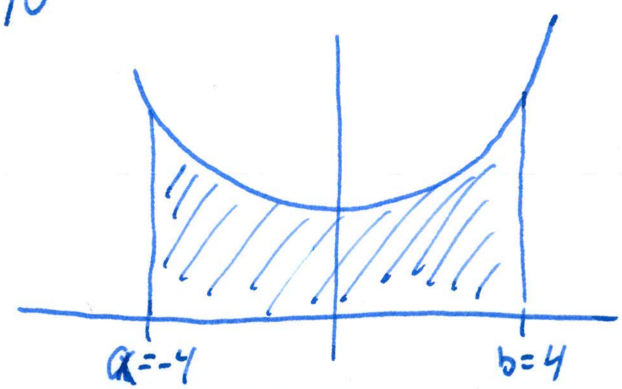
is <u>any</u> kind of Riemann Sum. (they all have the same limit.)

Complications

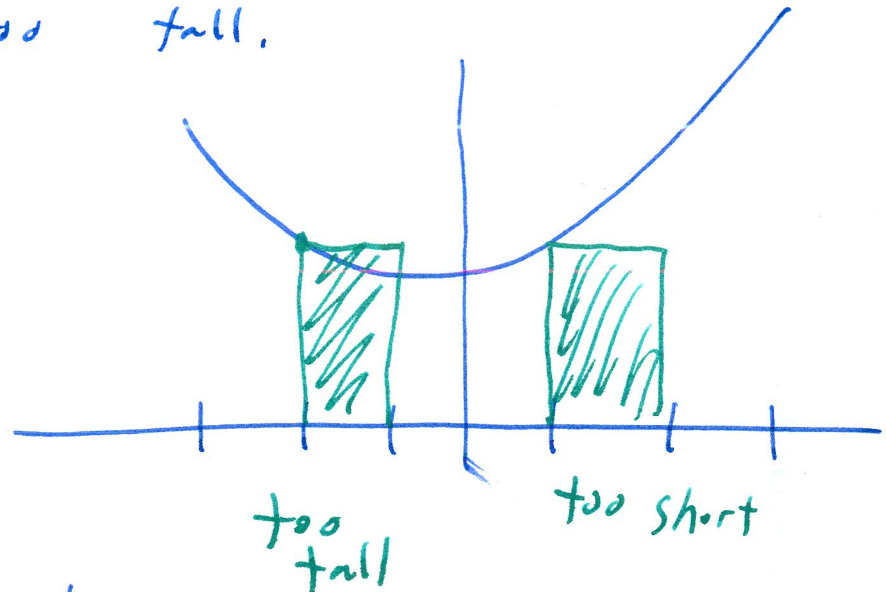
#1 What if the graph of f is <u>not</u> strictly increasing or decreasing?

Example $f(x) = 5 + \frac{x^2}{10}$ from $x=-4$ to $x=4$

$$A = \int_{x=-4}^{x=4} f(x)\,dx$$



Observe that some left rectangles are too short; Some are too tall.



So we don't know whether L_n will be less than, greater than, or equal to A.

The same thing is true of the right rectangles.

So we <u>cannot</u> say

$$L_n < A < R_n$$

But we can <u>still</u> define $A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

For example Let $f(x) = 5 + \frac{x^2}{10}$ Let $A = \int_{x=-4}^{4} f(x)\,dx$

Estimate A by using Riemann Sums.

$$L_8 = 44.4 = R_8$$

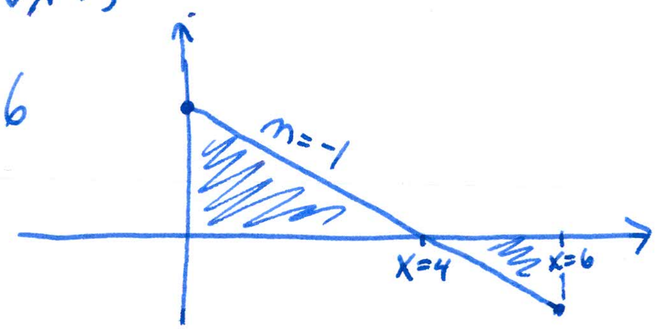
$$L_{100} \approx 44.2675 = R_{100}$$

$$L_{1000} = 44.2667 = R_{1000}$$

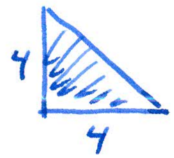
We could guess $A \approx 44.2667$

Complication #2 What if Graph of f dips below the x axis?

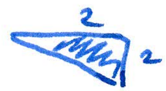
Example: $f(x) = -x + 4$ from $a=0$ to $b=6$



What are the areas of the shapes?

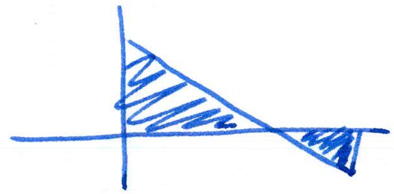


$\text{area} = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 4 \cdot 4 = 8$



$\text{area} = \frac{1}{2} b \cdot h = \frac{1}{2} \cdot 2 \cdot 2 = 2$

So total shaded area is



$= 8 + 2 = 10$

This is $\begin{pmatrix}\text{area of regions}\\ \text{above the x-axis}\end{pmatrix} + \begin{pmatrix}\text{area of regions}\\ \text{below x-axis}\end{pmatrix} = 8 + 2 = 10.$

But in Calculus, we're interested in what is called

the "Signed Area" $= \begin{pmatrix}\text{area of regions}\\ \text{above x-axis}\end{pmatrix} - \begin{pmatrix}\text{area of regions}\\ \text{below x-axis}\end{pmatrix}$

$= 8 - 2.$

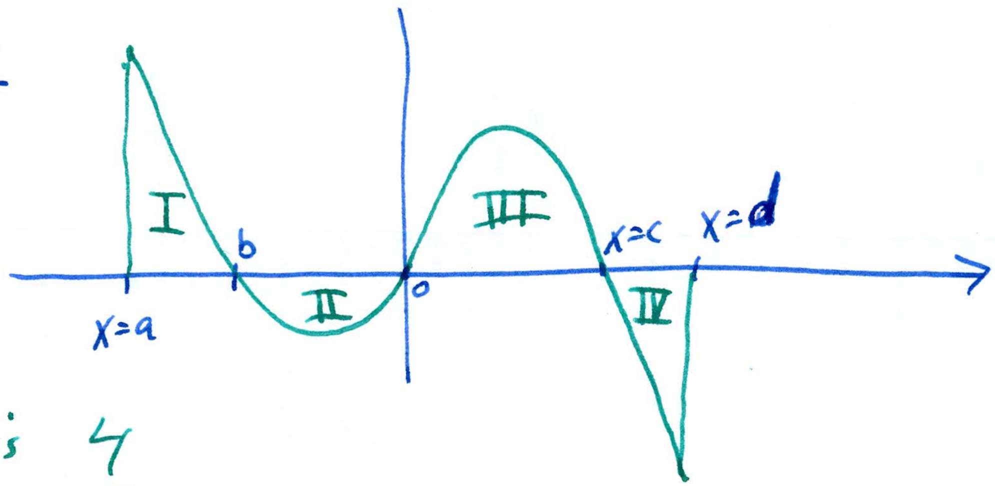
$= 6$ signed area.

The symbol for "signed area" is

$$A = \int_{x=a}^{x=b} f(x)\,dx$$

Examples of Signed Area

Graphical Example



Area of region I is 4

Area of region II is 3

Area of region III is 5

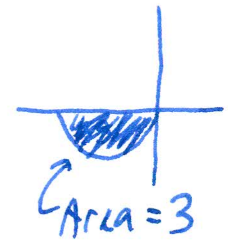
Area of region IV is 4

Example #1 Find $\int_{x=a}^{x=c} f(x)\,dx$

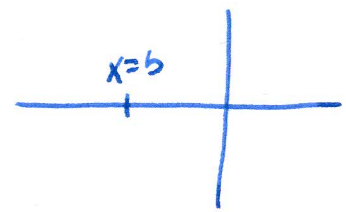
Solution $\text{I} - \text{II} + \text{III} = 4 - 3 + 5 = 6 = \int_{x=a}^{x=c} f(x)\,dx$

Example#2

$$\int_{x=b}^{x=0} f(x)\,dx = -\text{II} = -3$$



Example#3 $\int_{x=b}^{x=b} f(x)\,dx = 0$



Example#4

Find $\int_{x=d}^{x=a} f(x)\,dx$

This is weird. $x=d$ is not the left endpoint. It is the right endpoint. What do we do?

Solution: Page 389 Rule 2 tells us that the value of $\int_{x=d}^{x=a}$ is the negative of the integral $\int_{x=a}^{x=d}$

So $\displaystyle\int_{x=d}^{x=a} f(x)\,dx \underset{\text{Rule 2}}{=} -\int_{x=a}^{x=d} f(x)\,dx$

$$= -\left(\mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} \right)$$

$$= -(4 - 3 + 5 - 4)$$

$$= -(2)$$

$$= -2$$

~~Find Example~~ <u>Analytical Example</u>

Similar to suggested problem

Let $f(x) = 25 - 3x^2$

Partition the interval $[-4, 6]$ into five subintervals and find the Riemann sum using ~~five~~ rectangles whose heights are $f(c_k)$

where $c_k = \dfrac{x_{k-1} + x_k}{2}$

Solution

They are asking us to compute the midpoint sum M_5

Solution using Wolfram

$f(x) = 25 - 3x^2$

$a = -4$

$b = 6$

$n = 5$

midpoint sum $M_5 = -20$