2012-2013 Fall Semester MATH 1350 (Barsamian) In-Class Exam 3 Solutions

[**1**]($suggested\ exercise$ 5 − 1#37) $Given\ \ f(x) = (x - 5)e^{(-x)}$

(A) Find $f'(x)$ and make a sign chart for $f'(x)$. Be sure to show how the signs are determined.

$$\textbf{Solution}: f'(x) = \frac{d}{dx}\left((x-5)e^{(-x)}\right)$$
$$= \left(\frac{d}{dx}(x-5)\right)e^{(-x)} + (x-5)\left(\frac{d}{dx}e^{(-x)}\right)$$
$$= (1)e^{(-x)} + (x-5)\left((-1)e^{(-x)}\right)$$
$$= e^{(-x)} - xe^{(-x)} + 5e^{(-x)}$$
$$= e^{(-x)}(1 - x + 5)$$
$$= e^{(-x)}(6 - x)$$

Observe that $e^{(anything)}$ is always positive. The only thing in the factorization of f' that can change sign is the factor $(6 - x)$. It is zero only when $x = 6$. Choosing sample numbers to the left and right of $x = 6$, we can determine the sign of f'.

x-values	sample number	sign of f'	conclusion about f
$x < 6$	$x = 5$	$f'(5) = e^{(-(5))}(6 - (5)) = pos \cdot (1) = pos$	f is increasing
$x = 6$	$x = 6$	$f'(6) = e^{(-(6))}(6 - (6)) = pos \cdot (0) = 0$	f has horizontal tangent line
$6 < x$	$x = 7$	$f'(7) = e^{(-(7))}(6 - (7)) = pos \cdot (-1) = neg$	f is decreasing

(B) Using the information from your sign chart, find the intervals where f is increasing.

Solution: f is increasing on the interval $(-\infty, 6)$ because f' is positive there.

(C) Using the information from your sign chart, find the intervals where f is decreasing.

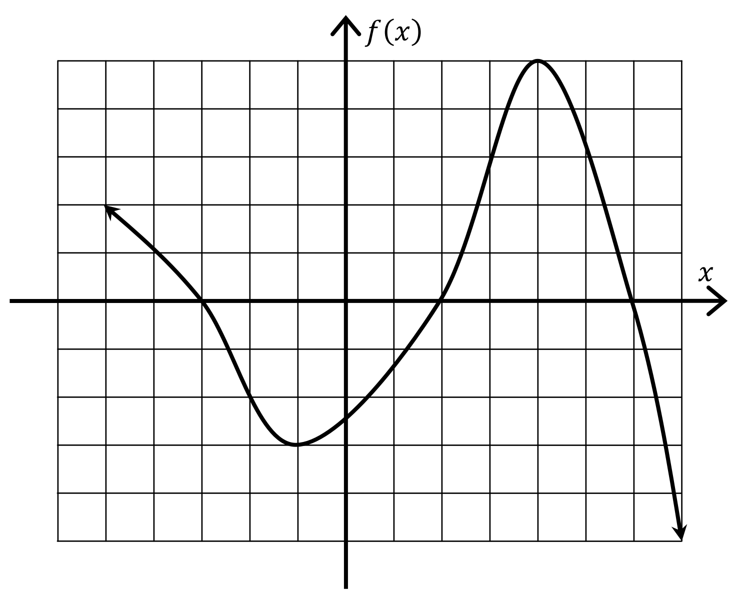
Solution: f is decreasing on the interval $(6, \infty)$ because f' is negative there.

(D) Using the information from your sign chart, find the *x*-values where f has a local max or min. (Be sure to say "max" or "min".)

Solution: f has a local max at $x = 6$ because f' changes from positive to zero to negative there.

[2] (based on Section 5-1 Exercise #78, #89 and a class drill)

Here is the graph of a function f.



(A) Find the values of x where $f(x) = 0$.

Solution: $f(x) = 0$ at $x = -3$ and $x = 2$ and $x = 6$ because its graph touches the x-axis there.

(B) Find the intervals where $f(x)$ is positive.

Solution: $f(x)$ is positive on intervals $(-\infty, -3)$ and $(2,6)$ because its graph is above the x-axis.

(C) Find the intervals where $f(x)$ is negative.

Solution: $f(x)$ is negative on intervals $(-3,2)$ and $(6, \infty)$ because its graph is below the x-axis.

(D) Find the values of x where $f'(x) = 0$.

Solution: $f'(x) = 0$ at $x = -1$ and $x = 4$ because its graph has a horizontal tangent line there.

(E) Find the intervals where $f'(x)$ is positive.

Solution: $f'(x)$ is positive on the interval $(-1,4)$ because its graph is increasing there.

(F) Find the intervals where $f'(x)$ is negative.

Solution: $f'(x)$ is positive on the intervals $(-\infty,-1)$ and $(4,\infty)$ because its graph is decreasing there.

[3]$(suggested\ exercise\ 5-2\#25)\ Let\ f(x)=\dfrac{x^4}{12}-\dfrac{5x^3}{6}+3x^2+23x+29$

Find the x-coordinates of all inflection points. Show your work clearly.

Solution: We need to find the x- coordinates of the points on the graph where the concavity changes. For that, we will study the sign of $f''(x)$. We first rewrite f in better form for taking the derivative. Then find f' and f''.

$$
\begin{aligned}
f(x) &= \left(\frac{1}{12}\right)x^4-\left(\frac{5}{6}\right)x^3+3x^2+23x+29\\
f'(x) &= \frac{d}{dx}\left(\left(\frac{1}{12}\right)x^4-\left(\frac{5}{6}\right)x^3+3x^2+23x+29\right)\\
&= \left(\frac{1}{12}\right)(4x^3)-\left(\frac{5}{6}\right)(3x^2)+3(2x)+23(1)+0\\
&= \left(\frac{1}{3}\right)x^3-\left(\frac{5}{2}\right)x^2+6x+23\\
f''(x) &= \frac{d}{dx}\left(\left(\frac{1}{3}\right)x^3-\left(\frac{5}{2}\right)x^2+6x+23\right)\\
&= \left(\frac{1}{3}\right)(3x^2)-\left(\frac{5}{2}\right)(2x)+6(1)\\
&= x^2-5x+6\\
&= (x-2)(x-3)
\end{aligned}
$$

We see that $f''(x)=0$ at $x=2$ and $x=3$. We suspect that these are the inflection points, but we have to confirm two things:

- We must confirm that $f(x)$ exists at those x-values, because there must be a point on the graph.
- We must confirm that the concavity changes at those x-values.

For the answer to the first question, we know that $f(x)$ exists at those x-values because f is a polynomial: its domain is all real numbers.

For the answer to the second question, we will investigate the sign of f'' at sample x-values

x-values	sample number	sign of f''	conclusion about f
$x<2$	$x=1$	$f''(1)=\big((1)-2\big)\big((1)-3\big)=neg\cdot neg=pos$	f is concave up
$x=2$	$x=2$	$f''(2)=\big((2)-2\big)\big((2)-3\big)=0\cdot neg=0$	
$2<x<3$	$x=5/2$	$f''(5/2)=\big((5/2)-2\big)\big((5/2)-3\big)=pos\cdot neg=neg$	f is concave down
$x=3$	$x=3$	$f''(3)=\big((3)-2\big)\big((3)-3\big)=pos\cdot 0=0$	
$3<x$	$x=4$	$f''(4)=\big((4)-2\big)\big((4)-3\big)=pos\cdot pos=pos$	f is concave up

We see that f'' does change sign at $x=2$ and $x=3$, so we conclude that f changes concavity at those x-values. Therefore, f has inflection points at $x=2$ and $x=3$.

[**4**]$(suggested\ exercise\ 5-5\#41)\ Given\ \ f(x)=8+x+\frac{9}{x}$
Find the absolute max and absolute min (if either of them exists) on the interval $(0,\infty)$.
Be sure to indicate if your result is a max or a min, and show how you know.
(You must use calculus and show all details clearly. No credit for just guessing *x*-values.)

Solution: Start by finding the critical values. Those are *x*-values that satisfy these two requirements:

- $f'(x)$ is undefined or $f'(x)=0$. (That is, *x* is a partition number for f'.)
- $f(x)$ exists.

We will need to find f'. First we rewrite f in a form that is better for taking the derivative. Then we find f'.

$$f(x)=8+x+9x^{-1}$$
$$f'(x)=\frac{d}{dx}(8+x+9x^{-1})$$
$$=0+1+9(-1)x^{-2}$$
$$=1-9x^{-2}=1-\frac{9}{x^2}$$

We see that $f'(x)$ is undefined at $x=0$. Next, we find *x*-values that cause $f'(x)=0$.

$$0=f'(x)$$
$$0=1-\frac{9}{x^2}$$
$$\frac{9}{x^2}=1$$
$$9=x^2$$

Two *x*-values make this equation true: $x=-3$ and $x=3$.
Therefore, f' has partition numbers $x=-3$ and $x=0$ and $x=3$.
Since $f(-3)$ exists and $f(3)$ exists while $f(0)$ does not exist, we conclude that only $x=-3$ and $x=3$ qualify to be called critical values of f. Of these two, only $x=3$ is in the interval $(0,\infty)$.

We need to determine if $x=3$ is the location of a max or a min or neither. There are two ways to do that:

Method #1: Study the sign of $f'(x)$ on the interval $(0,\infty)$.

x-values	sample number	sign of f'	conclusion about f
$0<x<3$	$x=1$	$f'(1)=1-\frac{9}{(1)^2}=1-9=neg$	f is decreasing
$x=3$	$x=3$	$f'(3)=1-\frac{9}{(3)^2}=1-1=0$	f has horizontal tangent line
$3<x$	$x=10$	$f'(10)=1-\frac{9}{(10)^2}=1-\frac{9}{100}=1-.09=pos$	f is increasing

We see that on the interval $(0,\infty)$ the graph of f is decreasing until $x=3$, then increasing after $x=3$. So the critical value $x=3$ is the location of the absolute min for the interval $(0,\infty)$. But remember the absolute min is the *y*-value, not the x-value. We substitute $x=3$ into $f(x)$ to get a y-value. The result: $f(3)=8+(3)+\frac{9}{(3)}=8+3+3=14$. Conclude that the absolute min is $y=14$. (It occurs at $x=3$.)

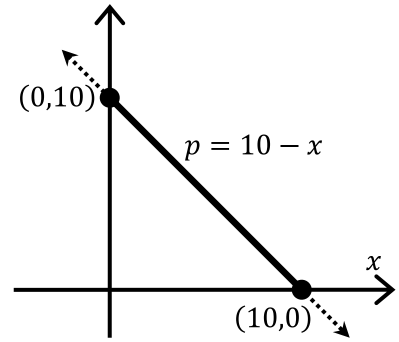
Method #2: Study the sign of $f''(x)$ the interval $(0,\infty)$.
So far, we have found that $f'(x)=1-9x^{-2}$. Therefore, $f''(x)=0-9(-2)x^{-3}=18x^{-3}=\frac{18}{x^3}$. We see that for all $x>0$, the value of $f''(x)$ will be positive. So the graph of f must be concave up for all $x>0$. So the critical value $x=3$ is the location of the absolute min for the interval $(0,\infty)$. As we did in the previous method, we find the value of the absolute min to be $y=14$.

[5] (suggested exercise 5-6#11) A company manufactures and sells hats.

- The demand x is the number of hats made each day.
- The price p is the selling price for each hat (in dollars).
- The daily price-demand equation is $p = 10 - x$.
- Remember that $Revenue = demand \cdot price = xp$.
- The Cost function is $C(x) = 7 + 2x$ (in dollars).
- Remember that $Profit = Revenue - Cost$.

(A) Graph the price-demand equation and find its domain. Explain how you know the domain.

Solution: The graph of the equation $p = 10 - x$ will be a line with slope $m = -1$. The vertical axis intercept will be at the point $(x, p) = (0,10)$; the horizontal axis intercept will be at the point $(x, p) = (10,0)$. If we considered the equation $p = 10 - x$ as an abstract mathematical equation, its domain would be the set of all real numbers x. But the equation is not just an abstract mathematical equation: it is modeling the making and selling of hats. Because one cannot make a negative number of hats, we know that x is restricted to $x \geq 0$. And because price cannot be negative, we know that $p \geq 0$. But in order to keep $p \geq 0$, we must have $x \leq 0$. Therefore, the domain is $0 \leq x \leq 10$.



(B) Find the Revenue function $R(x)$. (It should be a function of just the variable x, no p.)

Solution: $R(x) = xp = x(10 - x) = -x^2 + 10x$.

(C) If the goal is to maximize the daily Revenue, how many hats should be made each day, and what should be the selling price (in dollars) for each hat? Show all details clearly.

Solution: $R(x)$ is a continuous function defined on a closed interval $[0,10]$. So the closed interval method can be used to find the absolute max. We need to find the critical values of the function by finding $R'(x)$ and then setting $R'(x) = 0$ and solving for x. The derivative is $R'(x) = -2x + 10$. When we set $R'(x) = 0$ and solve for x, we find the critical value $x = 5$. Finally, we make a list of the important x-values and find their corresponding $R(x)$ values.

important x values	$R(x)$
$x = 0$ (endpoint)	$R(0) = 0(10 - 0) = 0$
$x = 5$ (critical)	$R(5) = 5(10 - 5) = 25$
$x = 10$ (endpoint)	$R(10) = 10(10 - 10) = 0$

We see that the maximum Revenue occurs when $x = 5$ hats are made each day. The corresponding selling price will be $p = 10 - 5 = 5$ dollars per hat.

(D) Find the Profit function $P(x)$. (It should be a function of the variable x.)

Solution: $P(x) = R(x) - C(x) = (-x^2 + 10x) - (7 + 2x) = -x^2 + 8x - 7$.

(E) If the goal is to maximize the daily Profit, how many hats should be made each day, and what should be the selling price (in dollars) for each hat?

Solution: We use the closed interval method again. We find the critical values of the function by setting $P'(x) = 0$ and solving for x. The derivative is $P'(x) = -2x + 8$. When we set $P'(x) = 0$ and solve for x, we find the critical value $x = 4$. Finally, we make a list of the important x-values and find their corresponding $P(x)$ values.

important x values	$P(x)$
$x = 0$ (endpoint)	$P(0) = -(0)^2 + 8(0) - 7 = -7$ (The company is losing \$7 per day!!)
$x = 4$ (critical)	$P(4) = -(4)^2 + 8(4) - 7 = -16 + 32 - 7 = 9$
$x = 10$ (endpoint)	$P(10) = -(10)^2 + 8(10) - 7 = -100 + 80 - 7 = -27$ (losing money)

We see that the maximum Profit occurs when $x = 4$ hats are made each day. The corresponding selling price will be $p = 10 - 4 = 6$ dollars per hat.