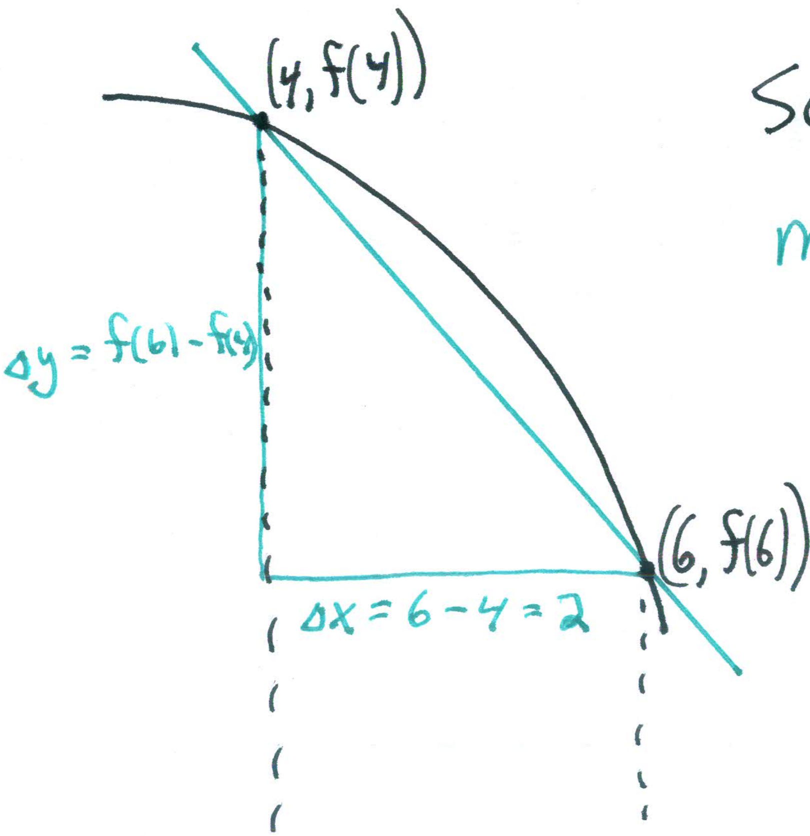
Day 7 is Tuesday, Jan 29, 2013

Yesterday Section 3-4 Rates of Change

Secant line connecting points where $x=4$ and $x=6$
coordinates of those points are $(x,y) = (4, f(4))$ and $(6, f(6))$



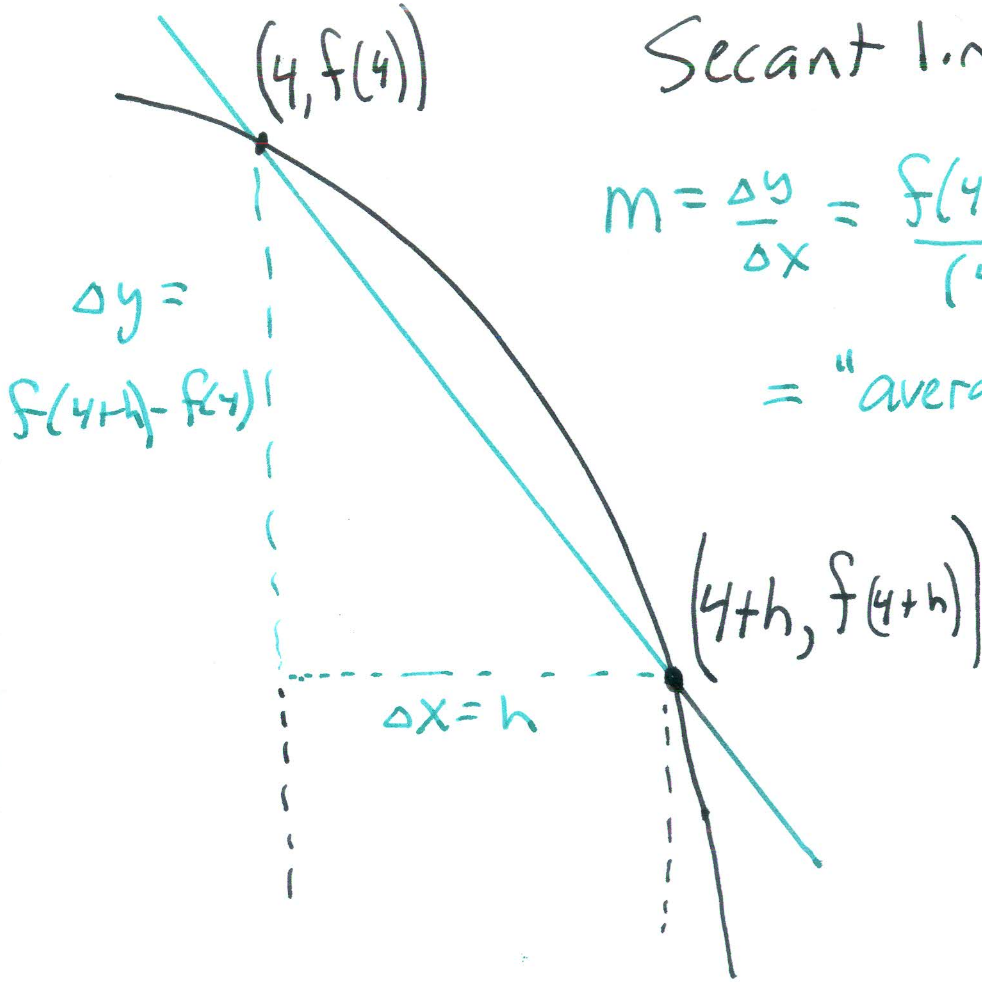
Secant line slope

$$m = \frac{\Delta y}{\Delta x} = \frac{f(6) - f(4)}{6 - 4} =$$

"average rate of change of f from 4 to 6."

Use 2^{nd} point with x-coordinate $4+h$

Two points are $(x,y)=(4,f(4))$ and $(x,y)=(4+h,\ f(4+h))$

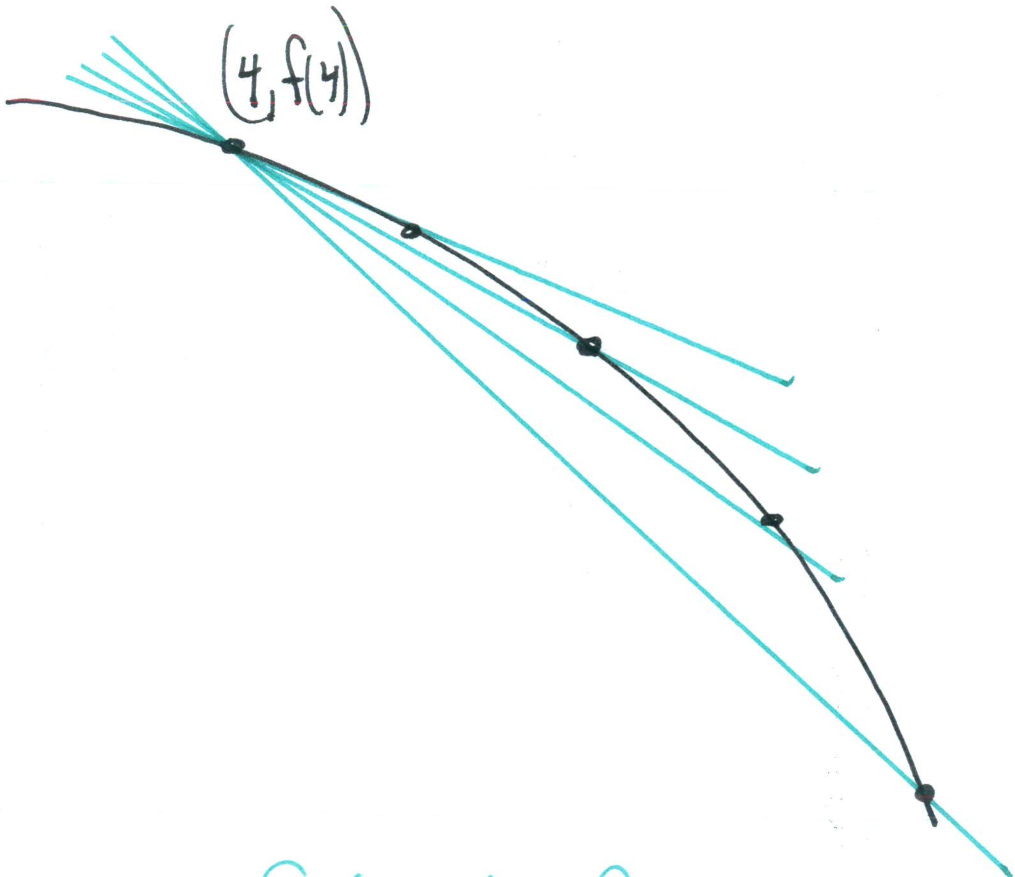


Secant line slope is

$$m=\frac{\Delta y}{\Delta x}=\frac{f(4+h)-f(4)}{(4+h)-4}=\frac{f(4+h)-f(4)}{h}$$

= "average rate of change of f from 4 to 4+h"

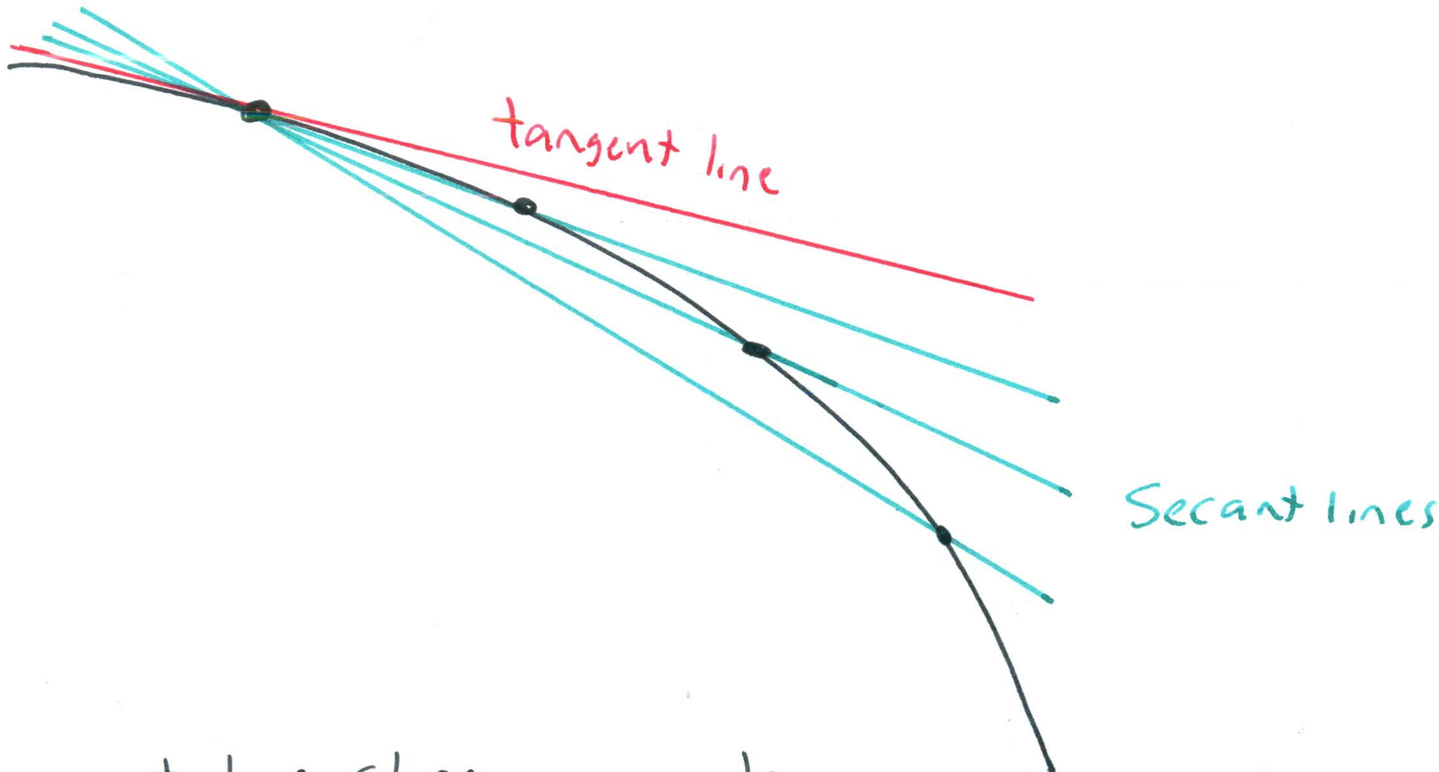
Imagine bring the second point closer to the first point (by making h smaller & smaller)



Slopes $m = \frac{f(4+h) - f(4)}{h}$ are getting less negative

secant line slope

What happens in the <u>limit</u>, as $h \to 0$?



$$\text{tangent line slope} = \lim_{h \to 0} \text{secant line slope}$$

$$m = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

The number m is also called the "Instantaneous rate of change of f at $x = 4$"

In general

- "Instantaneous Rate of change of f at a"

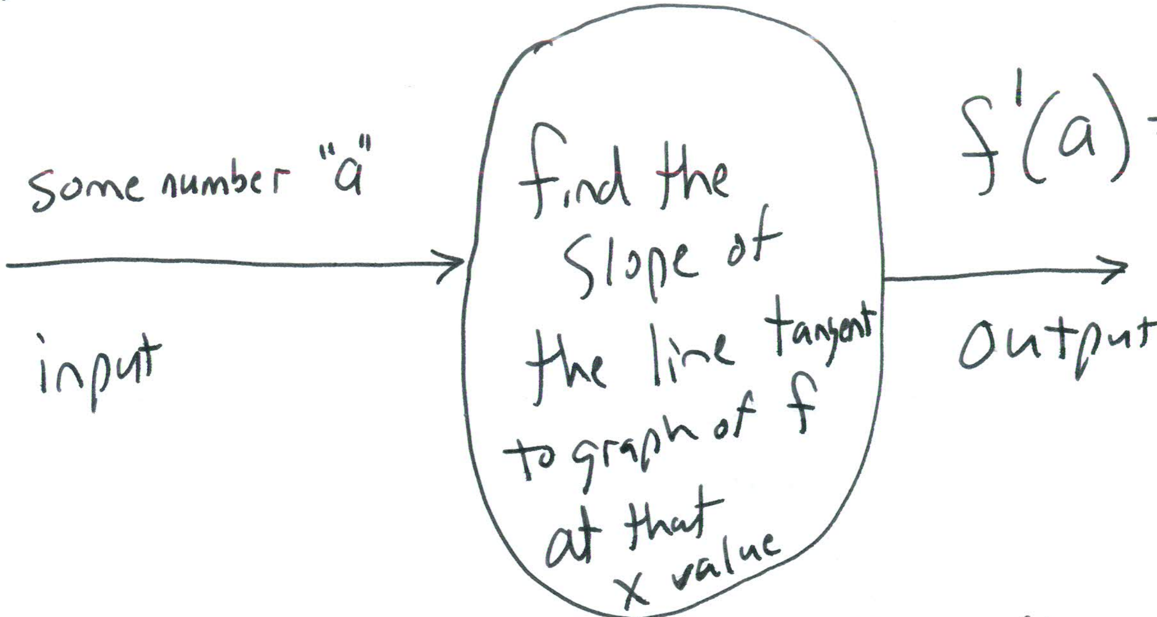
  means the number m that is the slope of the line tangent to graph of f at $x=a$

- m is computed by the formula

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

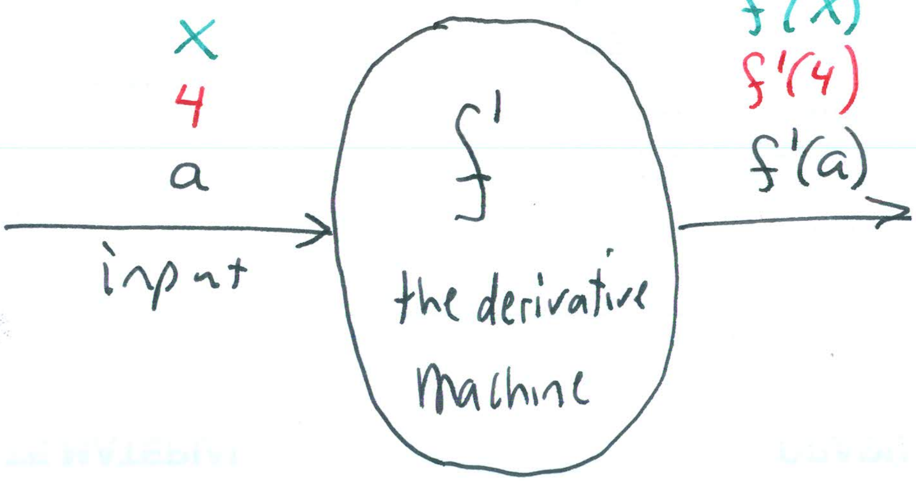
- also denoted $f'(a)$
- also called "the derivative of f at a"

Imagine this machine



$$f'(a) = m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This machine is really called the <u>derivative</u>



Examples of computing $f'(x)$

Example

$$f(x) = -3x^2 + 5x - 7$$

Find the derivative of f. That is, find $f'(x)$ using the definition of the derivative.

Solution

We have to build this

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

the definition of the derivative.

We have $f(x) = -3x^2 + 5x - 7$

get empty version $f(\ ) = -3(\ )^2 + 5(\ ) - 7$

$$f(x+h) = -3(x+h)^2 + 5(x+h) - 7.$$

$$= -3(x^2 + 2xh + h^2) + 5x + 5h - 7$$

$$= -3x^2 - 6xh - 3h^2 + 5x + 5h - 7$$

Get the numerator

$$f(x+h) - f(x) = \left(\cancel{-3x^2} - 6xh - 3h^2 + 5x + 5h - 7\right) - \left(\cancel{-3x^2} + 5x - 7\right)$$

these parentheses are extremely important

$$= -6xh - 3h^2 + 5h$$

factor out the h

$$= h(-6x - 3h + 5)$$

Build the quotient Remark: this is a "Difference Quotient" $\frac{\Delta y}{\Delta x}$

$$\frac{f(x+h) - f(x)}{h} = \frac{h(-6x - 3h + 5)}{h}$$

$$= -6x - 3h + 5$$

Finally, find $f'(x)$ by taking the limit as $h \to 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} -6x - 3h + 5 = -6x - 3(0) + 5 =$$

$\boxed{f'(x) = -6x + 5}$ the derivative of f.

Related questions

for the function $f(x) = -3x^2 + 5x - 7$

(A) Find the slope of the line that is tangent to the graph of f at $x=2$

Solution

Solution: we need to find $m = f'(2)$

We already know that $f'(x) = -6x + 5$

So we just substitute $x=2$ into that formula.

$m = f'(2) = -6(2) + 5 = -12 + 5 = -7$

(B) Find the slope of the line that is tangent to the graph of f at $x=0$.

Solution We need to find $m = f'(0)$

Substitute $x=0$ into formula for $f'(x)$

$$m = f'(0) = -6(0) + 5 = 5$$

$$m = 5$$

(C) Find the x-coordinates of all points on the graph of f that have horizontal tangent lines.

Solution

Horizontal lines have $m=0$. So we are trying to find all values of x that cause this:

$$m = f'(x) = 0$$

turn this around

$$0 = f'(x)$$
$$0 = -6x + 5$$

$$6x = 5$$

$$x = \frac{5}{6}$$

Harder example $f(x) = \frac{1}{x}$ Find $f'(x)$

Solution

Step 1 find $f(x+h)$

Solution

$$f(x) = \frac{1}{x}$$

$$f(\ ) = \frac{1}{(\ )} \quad \text{empty version}$$

$$f(x+h) = \frac{1}{(x+h)}$$

Step 2 find $f(x+h) - f(x)$

Solution $f(x+h) - f(x) = \dfrac{1}{x+h} - \dfrac{1}{x}$ get common denominator

$$= \frac{1}{(x+h)}\frac{x}{x} - \frac{1}{x}\frac{(x+h)}{(x+h)}$$

$$= \frac{x - (x+h)}{x(x+h)}$$

$$= \frac{-h}{x^2 + xh}$$

Step 3 Find the Difference Quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(f(x+h)-f(x)\right)$$

this layout makes it easier

$$= \frac{1}{h}\left(\frac{-h}{x^2+xh}\right)$$

$$= -\frac{1}{x^2+xh}$$

Step 4 find $f'(x)$ by taking the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} -\frac{1}{x^2+xh} = -\frac{1}{x^2+x(0)} = -\frac{1}{x^2}$$

$$f'(x) = -\frac{1}{x^2}$$

(B) Find the slope of the line tangent to the graph of $f(x) = \frac{1}{x}$ at $x=2$

Solution $m = f'(2) = \frac{-1}{(2)^2} = \frac{-1}{4}$

slope $= -1/4$

Now do a harder example involving a $\frac{1}{x}$ type function

Let $f(x) = 5 - \frac{17}{x}$ find $f'(x)$

Solution

Step 1 find $f(x+h)$

$$f(x) = 5 - \frac{17}{x}$$

$$f(\ \ ) = 5 - \frac{17}{(\ \ )}$$ empty version

$$f(x+h) = 5 - \frac{17}{(x+h)}$$

Step 2 find f(x+h) − f(x)

Solution

$$f(x+h) - f(x) = \left(5 - \frac{17}{(x+h)}\right) - \left(5 - \frac{17}{x}\right)$$

$$= \left(\frac{-17}{(x+h)}\right) - \left(\frac{-17}{x}\right)$$

factor out −17

$$= (-17)\left(\frac{1}{x+h}\right) - (-17)\left(\frac{1}{x+h}\right)$$

factor it out the rest of the way

$$= (-17)\left[\frac{1}{x+h} - \frac{1}{x}\right]$$

$$= -17\left[\frac{-h}{x^2+xh}\right]$$

using result from previous example (on page 12)

Step 3 Find the difference quotient

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(f(x+h) - f(x)\right)$$

$$= \frac{1}{h}\left[(-17)\left[\frac{-h}{x^2+xh}\right]\right]$$

$$= \frac{17}{x^2+xh}$$

Step 4 find $f'(x)$ by taking the limit

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{17}{x^2+xh} = \frac{17}{x^2+x(0)} = \frac{17}{x^2}$$