Day 35 is Thursday, April 11, 2013

Section 6-5 The Fundamental Theorem of Calculus

Intro:

We have seen two uses of the integral symbol $\int$.

(1) The indefinite integral having to do with antiderivatives.

$$F(x) = \int f(x)\,dx$$

- $F(x)$ is a function that is an antiderivative of $f(x)$.

  (that means that $F'(x) = f(x)$.)

(2) The Definite Integral having to do with signed area.

$$A = \int_{x=a}^{x=b} f(x)\,dx$$

A is a number that is the signed area under the graph of f from $x=a$ to $x=b$. (Defined as a limit of Riemann sums.)

So far it should seem strange that (1) and (2) use such similar-looking symbols. They seem to be unrelated concepts.

Here is the relationship between the two concepts.

The Fundamental Theorem of Calculus

If $F(x) = \int f(x)\,dx$ and $A = \int_{x=a}^{x=b} f(x)\,dx$

then $A = F(b) - F(a)$

Example

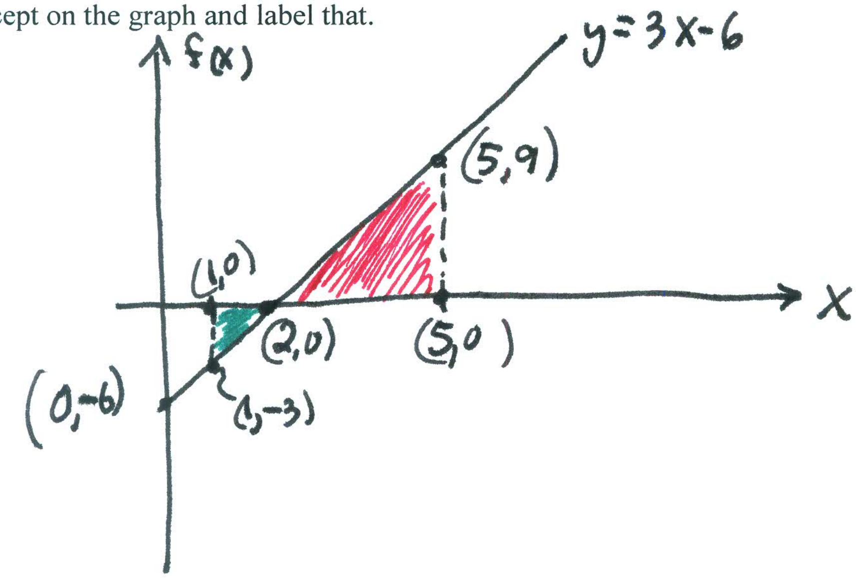
Class Drill 17 Finding Signed Area using two Different Methods

Class Drill 17: Finding the Signed Area Under a Graph By Two Different Methods

Part I: Drawing

Let $f(x) = 3x - 6$.

(a) Draw the graph of f for $0 \leq x \leq 6$. Make your graph large and neat. Find the coordinates of the x-intercept on the graph and label that.



(b) On your graph, shade the region between the graph of f and the x-axis from $x = 1$ to $x = 5$. The shaded region should be made up of two triangles.

Method #1 Finding Areas Using Geometry

(c) Using the geometric formula for the area of a triangle, find the area of each of the two triangles.

Green area $G = \frac{1}{2}(b)(h) = \frac{1}{2}(1)(3) = \frac{3}{2}$

Red area $R = \frac{1}{2}b(h) = \frac{1}{2}(3)(9) = \frac{27}{2}$

(d) Using the known areas of the two triangles,find the area of the shaded region. (The ***unsigned area***.) It should be the sum of the positive numbers that are the areas of the two triangles.

Unsigned area $USA = G + R = \frac{3}{2} + \frac{27}{2} = \frac{30}{2} = 15$

(e) Using the known areas of the two triangles, find the ***signed area*** of the shaded region. That is, find the value of

$$A = \int_{x=1}^{x=5} f(x)dx = \int_{x=1}^{x=5} 3x - 6dx$$

It should be the difference of the positive numbers that are the areas of the two triangles.

$$A = -G + R = -\frac{3}{2} + \frac{27}{2} = \frac{24}{2} = 12$$

Method #2 Finding The Signed Area Using Calculus

(f) Use the antiderivative formulas to find an antiderivative $F(x)$ for $f(x)$. That is, use the antiderivative formulas to find

$$F(x) = \int f(x)dx = \int 3x - 6dx$$

n=1 n=0

$$F(x) = \int 3x - 6dx = 3\int x\,dx - 6\int dx + C$$

$$= 3\left(\frac{x^2}{2}\right) - 6(x) + C$$

$$= \frac{3x^2}{2} - 6x + C$$

Check: Find $F'(x)$: $F'(x) = \frac{d}{dx}\left(\frac{3x^2}{2} - 6x + C\right) = \frac{3(2x)}{2} - 6(1) + 0$

$= 3x - 6$ ✓

(g) Check: Does $f'(x) = f(x)$? If not, then go back to step (f) and check your work.

(h) Using the function $F(x)$ that you found, compute $F(5) - F(1)$.

$$F(5) = \frac{3(5)^2}{2} - 6(5) + C = \frac{3 \cdot 25}{2} - 30 + C = \frac{75}{2} - 30 + C = \frac{15}{2} + C$$

$$F(1) = \frac{3(1)^2}{2} - 6(1) + C = \left(\frac{3}{2} - 6 + C\right) = \frac{-9}{2} + C$$

Comparing The Numbers Obtained By The Two Methods

$$F(5) - F(1) = \left(\frac{15}{2} + C\right) - \left(\frac{-9}{2} + C\right)$$

$$= \frac{24}{2} = 12$$

(i) Does your answer to question (e) match your answer to question (h)?

That is, does $A = F(5) - F(1)$?

That is, is the following equation true?

$$\int_{x=1}^{x=5} 3x - 6dx = F(5) - F(1)$$

12 = 12 ✓

from page (1) from page (2)

Examples using Fundamental Theorem of Calculus

Example 1 6-5 #12 (similar to suggested exercise #11)

Book: Find $\int_0^2 4e^x dx$

Solution

Name stuff and rewrite the problem.

Call our answer A. It will be a number, representing a signed area.

find $A = \int_{x=0}^{x=2} 4e^{(x)} dx$

Strategy: Find $F(x) = \int 4e^{(x)} dx$ and check!

Find $F(2)$ and $F(0)$

Compute $A = F(2) - F(0)$

$$F(x) = \int 4e^{(x)}dx = 4\int e^{(x)}dx = 4(e^{(x)}) + C$$

$$F(x) = 4e^{(x)} + C$$

check $F'(x) = \frac{d}{dx}\left(4e^{(x)} + C\right) = 4\frac{d}{dx}e^{(x)} + 0$

$$= 4e^{x}$$

$$F(2) = 4e^{(2)} + C$$

$$F(0) = 4e^{(0)} + C = 4 \cdot 1 + C = 4 + C$$

$$A = F(2) - F(0) = (4e^{2} + C) - (4 + C) = 4e^{2} - 4$$

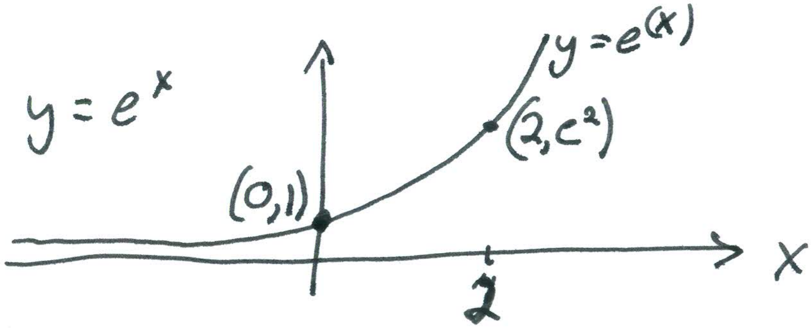
Make up a question (B) for this one

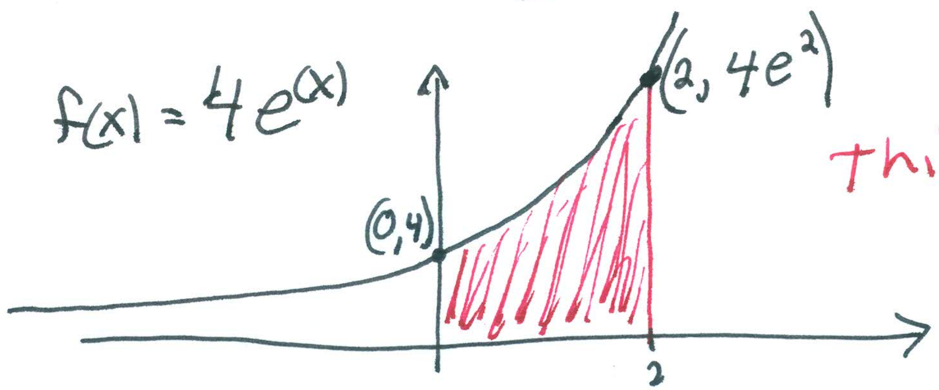
(B) Illustrate the answer from (A) using a graph.

Solution: Graph $f(x) = 4e^{(x)}$

little fcn. This is the integrand.

The quantity A can be illustrated by shading between graph of f and the x-axis.





This red ~~shaded~~ shaded region has area $A = 4e^2 - 4$

Example 2   6-5 #14

Find $A = \int_{x=1}^{x=5} \frac{2}{x}\,dx$ and illustrate with a graph.

Solution

Strategy: use fundamental theorem $A = F(b) - F(a)$

Find $F(x) = \int \frac{2}{x}\,dx = \int 2\left(\frac{1}{x}\right)dx = 2\int \frac{1}{x}\,dx$

$= 2\ln|x| + C$

$F(5) = 2\ln|5| + C = 2\ln(5) + C$

$F(1) = 2\ln|1| + C = 2\ln(1) + C = 2\cdot 0 + C = C$

$A = F(5) - F(1) = (2\ln(5) + C) - (C) = 2\ln(5)$

Hey! Wolfram says the answer is log(25). Why??

On the wolfram site, "log" means natural log.

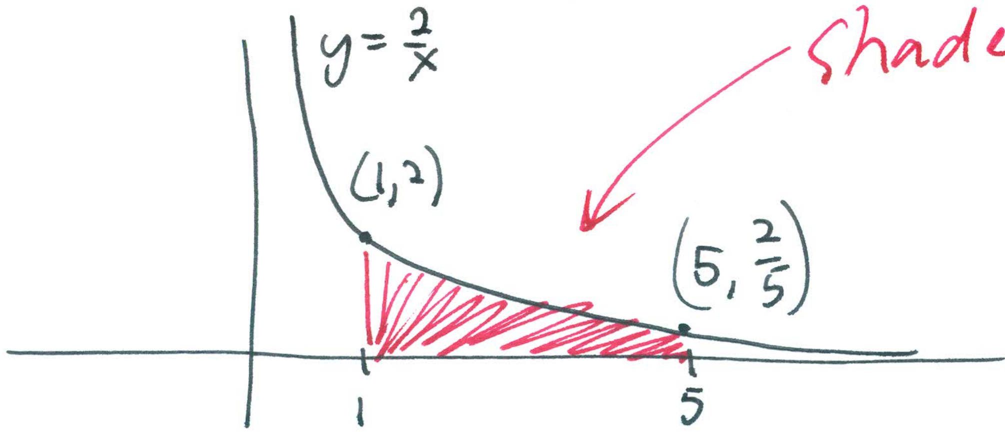
So wolfram's answer is $\ln(25)$

But $25 = 5^2$

So wolfram's answer is ~~ln(25)~~

$$\log(25) = \ln(25) = \ln(5^2) = 2\ln(5)$$

Illustrate with graph



shaded area is

$A = 2\ln(5)$