Monday, November 18, 2013 (Day 47)

- Swipe your I.D.
- Exam on Friday Covering Chapter 6
  - Bring your O.U. I.D.
- Start working on Class Drill 17

Resuming Section 6-5

The Fundamental Theorem of Calculus

$$\underbrace{\int_{x=a}^{x=b} f(x)\,dx}_{\text{the number that is the signed area under graph of } f \text{ from } x=a \text{ to } x=b} = \underbrace{F(b) - F(a)}_{\text{number obtained using } F(x) \text{ which is the antiderivative of } f(x),\ \text{obtained by } F(x) = \int f(x)\,dx}$$

Class Drill 17: The Fundamental Theorem of Calculus

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Given a continuous function f that is defined on a closed interval $a \le x \le b$, consider these two numbers that can be obtained.

The **first number** is the number $A = \int_{x=a}^{x=b} f(x)dx$. This number is called the *signed area* between the graph of f and the x-axis from $x = a$ to $x = b$. It can be visualized using a graph of f and in some simple cases it can be computed using geometry. But in general, the signed area is a *calculus* concept: it is defined as the limit of **Riemann sums**.

The **second number** is the number $F(b) - F(a)$, where F is an antiderivative of f. Notice that this second number also involves calculus, because it uses the antiderivative of f.

These two numbers seem to have nothing to do with one another. But the **Fundamental Theorem of Calculus** says that they are related to one another in a surprising way: they are equal!

In this class drill, you are given a continuous function f that is defined on a closed interval $a \le x \le b$. Your job will be to compute the first number (using geometry) and then compute the second number. Finally, you will compare the two numbers that you computed. They should turn out to be equal.

The given function is $f(x) = 2x - 4$ and the given interval is $1 \le x \le 5$.

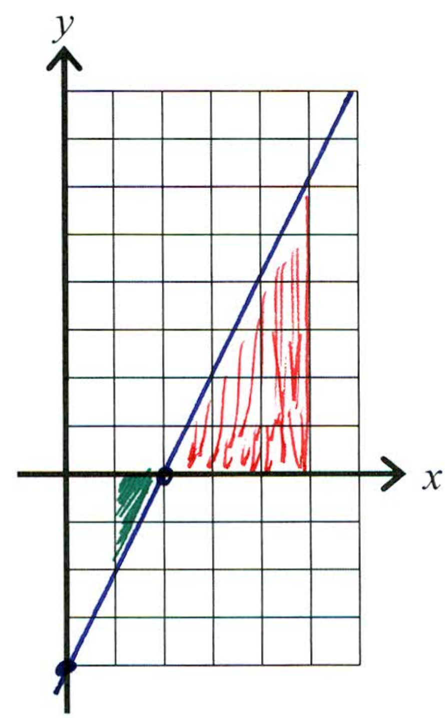
Part 1: Find the First Number

(a) Draw the graph of $f(x) = 2x - 4$ from $x = 0$ to $x = 6$.

$y = 2x - 4$

line with: x-intercept $(2,0)$

y-intercept $(0,-4)$



(b) On your graph, shade the region between the graph of f and the x-axis from $x = 1$ to $x = 5$. The shaded region should be made up of two triangles

(c) Using the geometric formula for the area of a triangle, find the area of each of the two triangles. The areas of triangles are positive numbers.

Green triangle area: $\frac{1}{2}bh = \frac{1}{2}(1)(2) = 1$

Red triangle area: $\frac{1}{2}bh = \frac{1}{2}(3)(6) = 9$

(d) Using the known areas of the two triangles, find the ***signed area*** of the shaded region. That is, find the value of

$$A = \int_{x=1}^{x=5} f(x)dx$$

It is obtained by first putting plus or minus signs in front of the positive numbers that are the areas of the two triangles, depending on whether the triangles are above or below the x-axis, and then adding the signed numbers together. The resulting number A is your **first number**.

Signed area $SA = -1 + 9 = 8$

Part 2: Find the Second Number

(e) Use the antiderivative formulas to find an antiderivative $F(x)$ for $f(x)$. That is, use the antiderivative formulas to find

$$F(x) = \int f(x)dx = \int 2x - 4dx$$

$n=1$ $n=0$

$F(x) = \int 2x - 4\,dx = 2\int x\,dx - 4\int (1)\,dx = 2\left(\frac{X^{1+1}}{1+1} + C_1\right) - 4\left(\frac{X^{0+1}}{0+1} + C_2\right)$

$= 2\left(\frac{X^2}{2} + C_1\right) - 4\left(\frac{X}{1} + C_2\right) = X^2 - 4X + \underline{2C_1 - 4C_2} = \boxed{X^2 - 4X + C_3}$

(f) Check: Does $F'(x) = f(x)$? If not, then go back to step (e) and check your work.

Check $F'(x) = \frac{d}{dx}(X^2 - 4X + C_3) = 2X - 4 + 0 = 2X - 4 = f(x)$ ✓

(g) Using the function $F(x)$ that you found in part (e), compute $F(5) - F(1)$. The resulting number is your **second number**.

$F(5) = ((5)^2 - 4(5) + C_3) = (25 - 20 + C_3) = 5 + C_3$

$F(1) = ((1)^2 - 4(1) + C_3) = (1 - 4 + C_3) = -3 + C_3$

$F(5) - F(1) = (5 + C_3) - (-3 + C_3) = 5 - (-3) = 8$

Part 3: Compare The Two Numbers

(h) Does your answer to question (d) match your answer to (g)? That is, is the following equation true?

$$\int_{x=1}^{x=5} f(x)dx = F(5) - F(1)$$

(The Fundamental Theorem of Calculus says that this equation is true.)

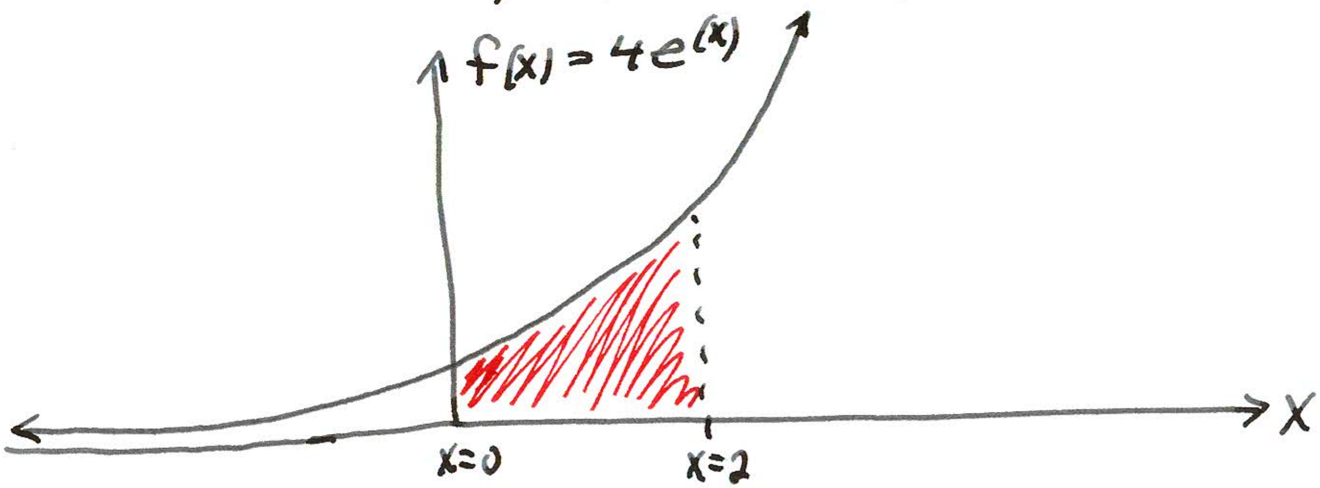
notice step (d) answer = step (g) answer

$8 = 8$

Examples Using the Fundamental Theorem of Calculus

Example #1 (exercise 6-5 #12) Find $\int_{x=0}^{x=2} 4e^{(x)}\,dx$.

Solution This symbol represents the signed area under the graph of $f(x) = 4e^{(x)}$ from $x=0$ to $x=2$.



We cannot find that signed area using geometry.

Instead, use Fundamental Theorem

$$SA = \int_{x=0}^{x=2} 4e^{(x)}\,dx = F(2) - F(0)$$

Strategy: Find the antiderivative $F(x)$

check it

Substitute in $x=0$ and $x=2$ + subtract

$$F(x) = \int 4e^{(x)}dx = 4\int e^{(x)}dx = 4\left(e^{(x)} + C_1\right) =$$

$$= 4e^{(x)} + \underbrace{4C_1}_{} = \boxed{4e^{(x)} + C_2}$$

Check: $F'(x) = \frac{d}{dx}\left(4e^{(x)} + C_2\right) = 4e^{(x)} + 0 = 4e^{(x)} = f(x)$ ✓

Substitute in numbers

$$F(2) = 4e^{(2)} + C_2 = 4e^2 + C_2$$

$$F(0) = 4e^{(0)} + C_2 = 4(1) + C_2 = 4 + C_2$$

Subtract

$$F(2) - F(0) = \left(4e^2 + C_2\right) - \left(4 + C_2\right) = 4e^2 - 4 = 4\left(e^2 - 1\right)$$

So our answer is

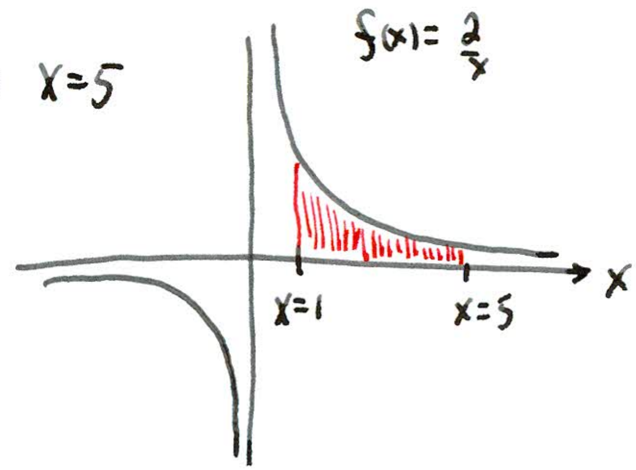
$$SA = \int_{x=0}^{x=2} 4e^{(x)}\,dx = F(2) - F(0) = 4(e^2 - 1) \text{ exact answer}$$

by Fundamental Theorem

≈ 25.6 approximate answer

Second Example 6-5 #14 Find $\displaystyle\int_{x=1}^{x=5} \frac{2}{x}\,dx$

Solution This symbol represents signed area under the graph of $f(x) = \frac{2}{x}$ from $x=1$ to $x=5$



We don't have a geometric formula for this.

So we will use the Fundamental Theorem

$$SA = \int_{x=1}^{x=5} \frac{2}{x}\,dx = F(5) - F(1)$$

Strategy: find $F(x)$

Check it

Substitute in $x=5$ & $x=1$ & subtract

$$F(x) = \int \frac{2}{x}\,dx = \int 2\left(\frac{1}{x}\right)dx = 2\int \frac{1}{x}\,dx = 2\left(\ln|x| + C_1\right)$$

$$= 2\ln|x| + 2C_1 = 2\ln|x| + C_2$$

Check $F'(x) = \frac{d}{dx}\left(2\ln|x| + C_2\right) = 2\frac{d}{dx}\ln|x| + 0$

$$= 2\left(\frac{1}{x}\right) = \frac{2}{x} = f(x) \checkmark$$

Now substitute in $x=5$ & $x=1$

$$SA = F(5) - F(1) = (2\ln|5| + C_2) - (2\ln|1| + C_2)$$

$$= 2\ln|5| - 2\ln|1|$$

$$= 2(\ln|5| - \ln|1|)$$

$$= 2(\ln(5) - \ln(1))$$

because $|5| = 5$, $|1| = 1$

$$= 2(\ln(5) - 0)$$

remember $e^{(0)} = 1$

$0 = \ln(1)$

$$= 2\ln(5)$$

Computer gives answer

$\log(25)$ But in Wolfram Alpha $\log(x)$ means $\ln(x)$

$\log(25) = \ln(25) = \ln(5^2) = 2\ln(5)$ because $\ln(a^b) = b\ln(a)$

Example#3 Similar to 6-5 #15

Find $SA = \int_{x=-3}^{x=3} x^3 + 5x\,dx$

Solution

Use Fundamental Theorem

$$SA = \int_{x=-3}^{x=3} x^3 + 5x\,dx = F(3) - F(-3)$$

Strategy: Find $F(x)$, check it, then compute $F(3) - F(-3)$

$$F(x) = \int x^3 + 5x\,dx = \frac{x^4}{4} + \frac{5x^2}{2} + C$$

Check: $F'(x) = \frac{d}{dx}\left(\frac{x^4}{4} + \frac{5x^2}{2} + C\right) = \frac{4x^3}{4} + \frac{5(2x)}{2} + 0$

$= x^3 + 5x = f(x)$ ✓ good. our $F(x)$ is correct

Now Find $F(3) - F(-3)$

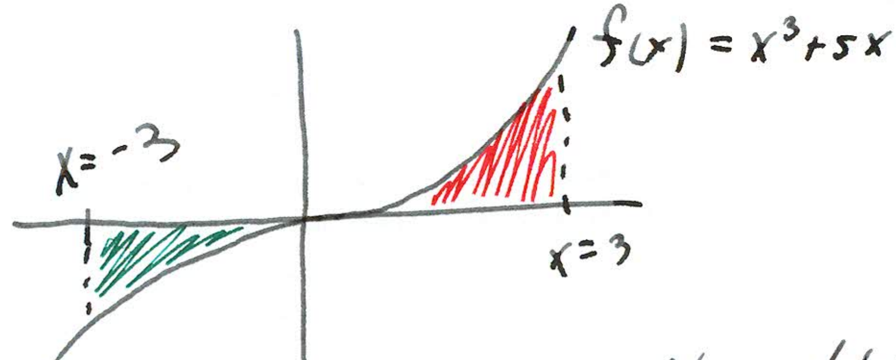
$$SA = F(3) - F(-3)$$

$$= \left(\frac{(3)^4}{4} + \frac{5(3)^2}{2} + C\right) - \left(\frac{(-3)^4}{4} + \frac{5(-3)^2}{2} + C\right)$$

$$= \cancel{\left(\frac{81}{4} + \frac{5(9)}{2}\right) - \left(\frac{81}{4} + \frac{5(9)}{2}\right)}$$

$$= 0 \quad ?!?$$ Why did this happen?

This is an odd function. (The left side is the upside down mirror image of the right side.) So the signed areas cancel.



So it turns out that this is a problem where we could have determined the signed area directly, by considering the shape of the graph.