Tuesday, Nov 19, 2013 (Day 48)

Swipe your I.D.

Pick up your graded work

Exam Friday Covers Chapter 6
(Bring your I.D. (Sorry!))

Study the Substitution Method!!

Continuing the Fundamental Theorem of Calculus today, with applications of the fundamental theorem.

- Total Change Problems
- Average value of a function over an integral.

Total Change Problems

The form of the fundamental theorem that we have been using so far is

$$\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$$

In this expression, $f(x)$ is the integrand.

$F(x)$ is an antiderivative obtained by doing an indefinite integral.

$$\begin{tikzcd} F(x) \arrow[rr, bend left, "\frac{d}{dx}F(x)" above, "\text{Check}" below] & & f(x) \arrow[ll, bend left, "\text{Find } F" above, "\int f(x)\,dx" below] \end{tikzcd}$$

Another form

use integrand $F'(x)$ in the Fundamental Theorem

$$\int_{x=a}^{x=b} F'(x)\,dx = F(b) - F(a)$$

$$\begin{tikzcd}
F(x) \arrow[rr, bend left, "\frac{d}{dx}F(x)", "\text{check}"'] & & F'(x) \arrow[ll, bend left, "\text{find } F(x)"', "\int F'(x)\,dx"]
\end{tikzcd}$$

integrand

"Total Change" refers to the ~~the~~ idea that the expression $F(b) - F(a)$ represents a change in F. (the "total change")

total change $= \Delta F = F(b) - F(a)$.

We will be doing problems where the stated goal is to find some total change ΔF.

That will lead us to doing the definite ~~fact~~ integral.

(Our previous uses of the Fundamental Theorem were in problems where we wanted to find a signed area.)

Example#1 of a Total Change Problem

A company makes & sells bikes.

Demand x is the number of bikes made in a month.
(units: bikes per month)

Cost $C(x)$ is the cost of producing that monthly batch of bikes (units: dollars per month)

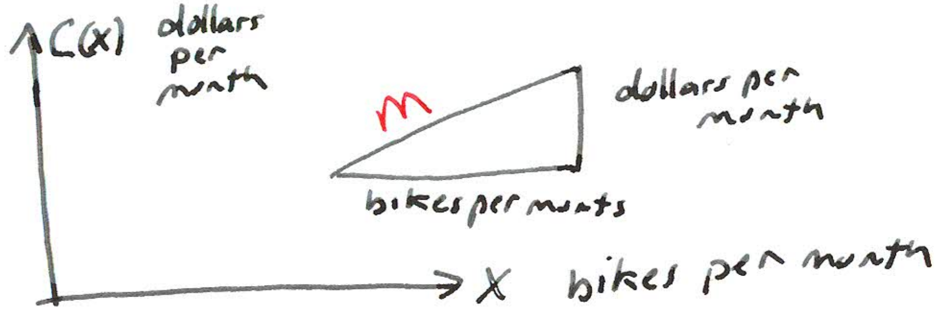
We don't know what $C(x)$ is.

Suppose we are given the marginal cost function

$$C'(x) = 1000 - \frac{x}{2} \quad \text{for} \quad 0 \le x \le 800$$

Remark

The units of C' ought to be



The slope m would have units

$$\frac{\text{dollars per month}}{\text{bikes per month}} = \frac{\text{dollars}}{\text{bike}} \quad \text{units of } C'$$

Note that the book says the units of C' are dollars.

We will therefore ignore the issue of units of C'.

Question: What will be the change in cost of a batch of bikes if the batch size is changed from 200 bikes/month to 400 bikes/month?

That is, what is the total change

$$\Delta C = C(400) - C(200) ?$$

Solution ~~Step~~ We will use the Fundamental Theorem

$$\Delta C = C(400) - C(200) \underset{\substack{\uparrow \\ \text{Fundamental} \\ \text{Theorem}}}{=} \int_{x=200}^{x=400} C'(x)\,dx = \int_{x=200}^{x=400} 1000 - \frac{x}{2}\,dx$$

Strategy:
- Find $C(x)$ using indefinite integral $\int C'(x)\,dx$.
- check
- use $C(x)$ to compute ΔC.

$$C(x) = \int C'(x)\,dx = \int 1000 - \frac{x}{2}\,dx$$

$n=0$ because $1 = x^0$ $\quad$ $n=1$ because $x = x^1$

$$= 1000 \int 1\,dx - \left(\frac{1}{2}\right) \int x\,dx$$

$$= 1000\left(\frac{x^{0+1}}{0+1} + K_1\right) - \frac{1}{2}\left(\frac{x^{1+1}}{1+1} + K_2\right)$$

$$= 1000x - \frac{x^2}{4} + \underbrace{1000K_1 - \left(\tfrac{1}{2}\right)K_2}_{\text{this could be any real number}}$$

$$C(x) = 1000x - \frac{x^2}{4} + K_3$$

This is almost the cost function, but not completely because we don't know what K_3 is. But we won't need K_3 so we go on.

Check by differentiating

$$C'(x) = \frac{d}{dx}\left(1000x - \frac{x^2}{4} + K_2\right)$$

$$= 1000\frac{dx}{dx} - \frac{1}{4}\frac{d}{dx}x^2 + 0$$

$$= 1000(1) - \left(\frac{1}{4}\right)(2x) + 0$$

$$= 1000 - \frac{x}{2} = C'(x)$$

Now compute ΔC

$$\Delta C = C(400) - C(200) = \cancel{\left(1000 - \frac{400}{2} + K_3\right)}$$

I used C'!! Bad!

$$= \left(1000(400) - \frac{(400)^2}{4} + K_3\right) - \left(1000(200) - \frac{(200)^2}{4} + K_3\right)$$

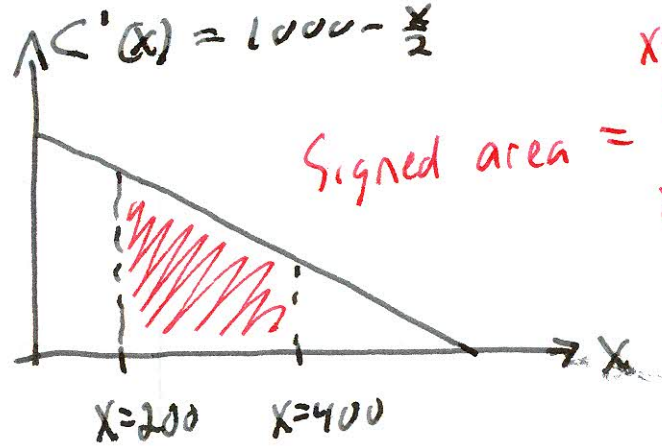
$$= 400000 - \frac{160000}{4} - 200000 + \frac{40000}{4}$$

$= 400000 - 40000 - 200000 + 10000$

$\Delta C = 120{,}000$

The change in cost of a monthly batch of bikes will be 120,000.

Picture of this



Signed area $= \int_{x=200}^{x=400} C'(x)\,dx$

Another Total Change Example

Bacteria Culture problem

t is time in hours

$W(t)$ is the weight (in grams) of a bacteria culture.

We're given that the bacteria is growing at a rate

$$W'(t) = 0.6\, e^{(0.2t)} \quad \frac{\text{grams}}{\text{hour}}$$

How much does the weight change from $t=5$ to $t=15$ hours?

That is, find $\Delta W = W(15) - W(5)$

Background problem

Find $\int e^{(kx)}\,dx$

Solution Use u-substitution

integrand $f(x) = e^{(kx)}$

Step 1 identify $u(x) = kx$ = inner function

Step 2+3 find $\frac{du}{dx}$ and solve for dx

$$u = kx$$

$$\frac{du}{dx} = k$$

multiply by dx and divide by k

$$\frac{1}{k}\,du = dx$$

Steps 4+5 Substitute + simplify

$$\int e^{kx}\,dx \xrightarrow[\text{substitute}]{} \int e^{u}\left(\frac{1}{k}\right)du = \left(\frac{1}{k}\right)\int e^{u}\,du$$

Step 6 find the indefinite integral

$$\left(\frac{1}{k}\right)\int e^u\,du = \left(\frac{1}{k}\right)\left(e^u + C\right) = \frac{e^u}{k} + \frac{C}{k} = \frac{e^u}{k} + C_2$$

this can be any constant

Step 7 Substitute back to get F(x)

$$\frac{e^u}{k} + C_2 \xrightarrow[\text{Substitute}]{} \frac{e^{kx}}{k} + C_2$$

Result $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C_2 = F(x)$

Step 8 check

$$F'(x) = \frac{d}{dx}\left(\frac{e^{kx}}{k} + C_2\right) = \frac{1}{k}\frac{de^{kx}}{dx} + 0 = \left(\frac{1}{k}\right)\left(ke^{kx}\right) = e^{kx} = f(x)$$

used derivative rule $\frac{d}{dx}e^{kx} = ke^{kx}$

Tomorrow, we will use this result

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$

to solve the total change problem