Monday, November 25, 2013 (Day 50)

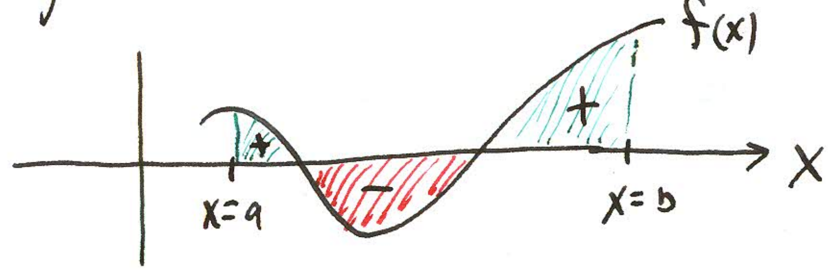
- Swipe your I.D.

Chapter 7 The Area Between Two Curves

Recall the Definition of

The Signed Area Under Graph of f from $x=a$ to $x=b$.

Picture

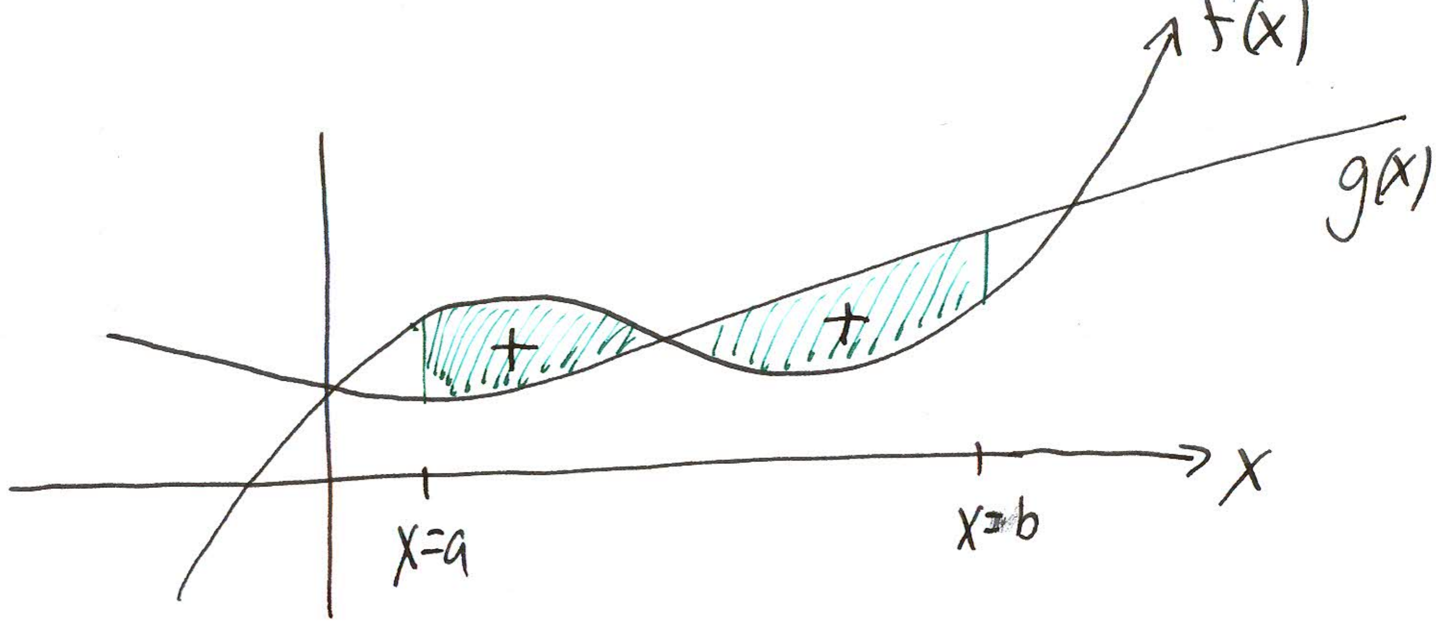


calculation: $SA = \int_{x=a}^{x=b} f(x)\,dx$ This definite integral takes care of the signs.

New Definition

"The area between graphs of $f(x)$ and $g(x)$ from $x=a$ to $x=b$."

Picture



All regions get plus signs. The "area between graphs of $f(x)$ and $g(x)$" means unsigned area

Question: How do we compute this area?

Answer in simple case where one of the graphs stays on top.

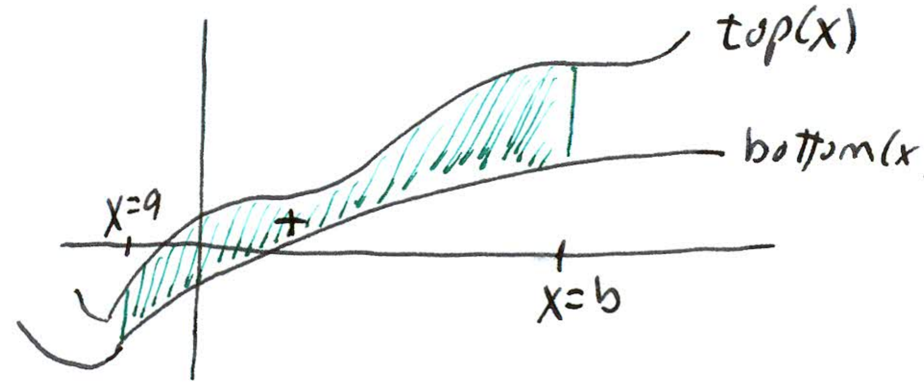
Theorem 1

If top(x) and bottom(x) are two functions that

- are continuous on the interval $a \le x \le b$
- have property that $\text{bottom}(x) \le \text{top}(x)$ for $a \le x \le b$

Then the "area between graph of top(x) and bottom(x) from $x=a$ to $x=b$" is equal to the number obtained by doing this definite integral

$$USA = \int_{x=a}^{x=b} \text{top}(x) - \text{bottom}(x)\,dx$$



In practice, the definite integral is fairly easy to do. (we did Definite Integrals in Ch.6) What is difficult is determining in a given problem whether or not we have a top(x) function and a bottom(x) function.

Example#1

Find the area between the graphs of $y = 2x - 2$ and $y = x^2 + 1$ from $x = -3$ to $x = 2$.

Solution: We would like to solve this by doing an integral $USA = \int_{x=-3}^{x=2} top(x) - bottom(x)\,dx$.

In order to do that, we ~~must~~ must determine if we have a top graph & bottom graph.

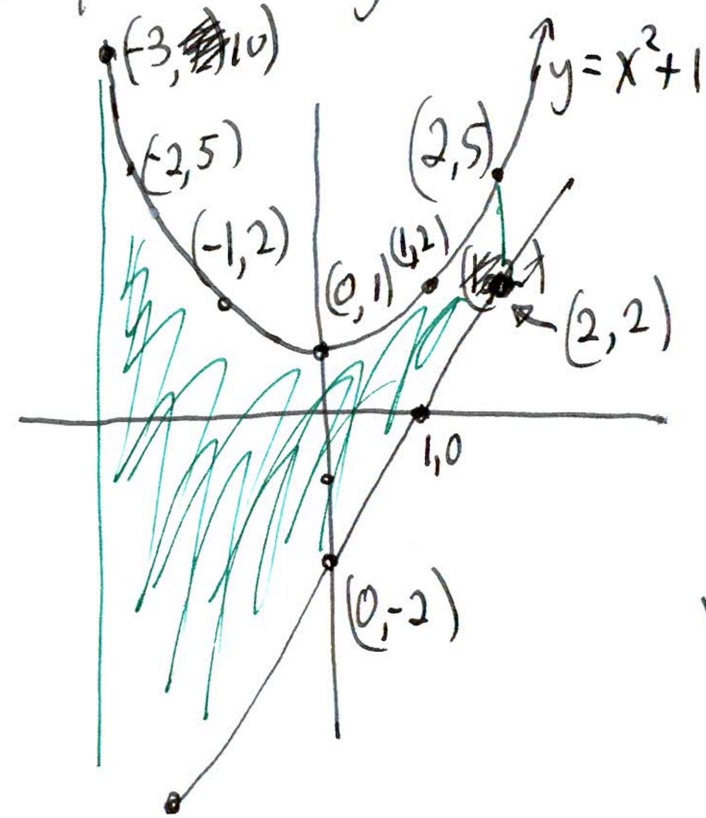
Graph of $y = 2x - 2$ is a straight line

Slope $m = 2$

y-intercept at $(x, y) = (0, -2)$

Graph of $y = x^2 + 1$ is a standard parabola moved up one unit

So vertex at $(x, y) = (0, 1)$



Clearly, we can use

$$top(x) = x^2 + 1$$

$$bottom(x) = 2x - 2$$

So we <u>can</u> use theorem 1

$$USA = \int_{x=-3}^{x=2} top(x) - bottom(x)\, dx$$

$$= \int_{x=-3}^{x=2} \underbrace{(x^2+1) - (2x-2)}_{\text{simplify before integrating !!!}}\, dx$$

$$= \int_{x=-3}^{x=2} x^2 - 2x + 3 \, dx$$

$$= \left(\overbrace{\frac{x^3}{3} - \frac{2x^2}{2} + 3x + C}^{\text{general antiderivative}} \right) \Bigg|_{x=-3}^{x=2}$$

New notation for the definite integral

this means plug in $x=2$ and $x=-3$ and subtract

In Ch. 6, we would have written $F(2)-F(-3)$

$$USA = \left(\frac{x^3}{2} - x^2 + 3x + C \right) \Bigg|_{x=-3}^{x=2}$$

$$= \underbrace{\left(\frac{2^3}{3} - 2^2 + 3(2) + C \right)}_{F(b)} - \underbrace{\left(\frac{(-3)^3}{3} - (-3)^2 + 3(-3) + C \right)}_{F(a)}$$

Chapter 6 Notation would say this

$$= \frac{8}{3} - 4 + 6 - (-9 - 9 - 9)$$

$$= \frac{8}{3} - 4 + 6 + 9 + 9 + 9$$

$$= \frac{8}{3} + 29 = \frac{8}{3} + \frac{87}{3} = \frac{95}{3}$$

Theorem #1 Does not say anything about the situation where the graphs cross in the interval $a \leq x \leq b$. If the graphs of f + g cross, you have to figure out the x-values where they cross.

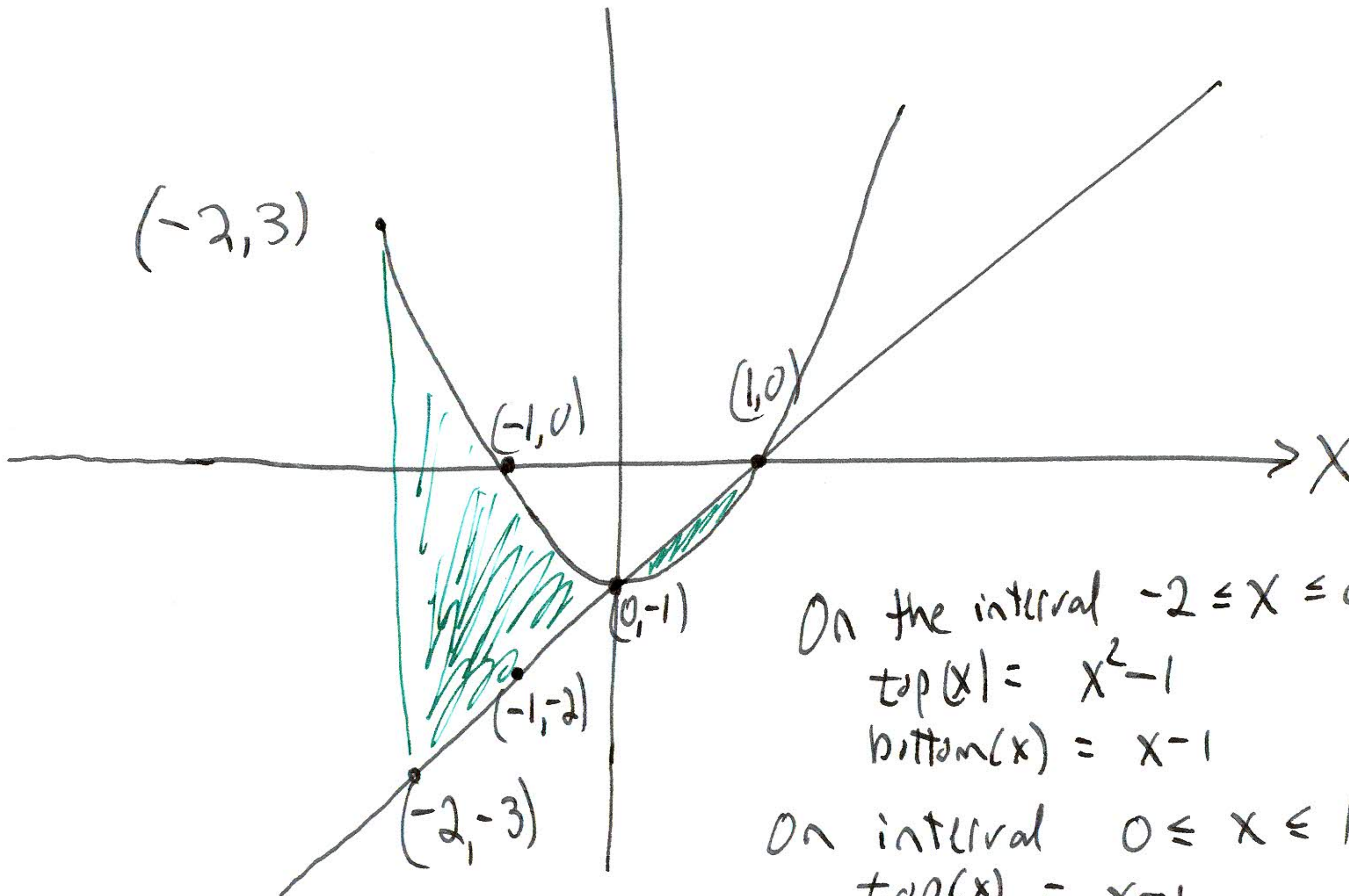
This breaks the domain up into intervals where one graph plays the role of the "top" graph. On those ~~sub~~ intervals, you do a simple definite integral

$$\int_{x_1}^{x_2} top(x) - bottom(x)\, dx$$

Knowing that "top" + "bottom" changes on each interval.

Example Find the are between the graphs of $y = x^2 - 1$ and $y = x - 1$ from $x = -2$ to $x = 1$.

Solution We must figure out if the graphs cross.



On the interval $-2 \le x \le 0$

$\text{top}(x) = x^2 - 1$

$\text{bottom}(x) = x - 1$

On interval $0 \le x \le 1$

$\text{top}(x) = x - 1$

$\text{bottom}(x) = x^2 - 1$

So we must do <u>two</u> definite integrals

$$USA = \int_{x=-2}^{x=0} \underbrace{(x^2-1)-(x-1)}_{\text{Simplify before integrating!}}\,dx + \int_{x=0}^{x=1} \underbrace{(x-1)-(x^2-1)}_{\text{Simplify before integrating!}}\,dx$$

$$= \int_{x=-2}^{x=0} x^2 - x\,dx + \int_{x=0}^{x=1} x - x^2\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{x=-2}^{x=0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=1}$$

$$= \underbrace{\cancel{\left(\frac{0^3}{3} - \frac{0^2}{2}\right)}}_{\text{zero}} - \left(\frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right) + \left(\frac{1^2}{2} - \frac{(1)^3}{3}\right) - \underbrace{\cancel{\left(\frac{0^2}{2} - \frac{0^3}{3}\right)}}_{\text{zero}}$$

$$= \frac{8}{3} + \frac{4}{2} + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{7}{3} + 2 + \frac{1}{2}$$

$$= \frac{7}{3} + 2 + \frac{1}{2}$$

$$= \frac{14}{6} + \frac{12}{6} + \frac{3}{6}$$

$$= \frac{29}{6}$$