Solutions for 2013-2014 Spring Semester MATH 3210/5210 (Barsamian) Homework 7
(Due Friday, April 4, 2014)

[1] (20 points) (Five True False questions from suggested exercise 4.4#1.) The True/False answers for some of the questions are given in the back of the book. For each question, give the true/false answer and also explain why the answer is true or false. Justify your explanation by referring to things that the book says in Section 4.4

4.4#1 (c) If two rows or columns of a matrix A are identical, then $det(A) = 0$.	**TRUE This is Property 5 of the Determinant, page 235**
4.4#1 (g) The determinant of an upper-triangular $n \times n$ matrix is the product of its diagonal entries.	**TRUE This is Property 4 of the Determinant, page 235**
4.4#1 (i) If $A, B \in M_{n\times n}(F)$, then $det(AB) = det(A)det(B)$.	**TRUE This is Property 6 of the Determinant, page 235**
4.4#1 (j) If Q is an invertible matrix, then $det(Q^{-1}) = \left(det(A)\right)^{-1}$.	**TRUE This is Property 7 of the Determinant, page 236**
4.4#1 (k) A matrix Q is invertible if and only if $det(Q) \neq 0$.	**TRUE This is also Property 7 of the Determinant, page 236**

[2] (20 points) (Similar to suggested exercise 4.4#2(b),(d))
Evaluate the determinant of the following matrices. Show all details of the calculations.

(a) $A = \begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix}$ **Solution:** $\boldsymbol{det(A) = det\begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix} = 3(2) - 1(-5) = 11.}$

(b) $B = \begin{pmatrix} 5i & 7i \\ -3 & 2i \end{pmatrix}$ **Solution:** $\boldsymbol{det(B) = det\begin{pmatrix} 5i & 7i \\ -3 & 2i \end{pmatrix} = (5i)(2i) - (-3)(7i) = -10 + 21i.}$

[3] (20 points) (Similar to suggested exercise 4.4#3(d)) Evaluate the determinant along the third row. Show all details of the calculation.

$$A = \begin{pmatrix} 2 & 0 & 7 \\ 3 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\boldsymbol{Solution}: \boldsymbol{det(A) = (-1)^{3+1}A_{31}det(\bar{A}_{31}) + (-1)^{3+2}A_{32}det(\bar{A}_{32}) + (-1)^{3+3}A_{33}det(\bar{A}_{33})}$$
$$\boldsymbol{= (-1)^4(0)det\begin{pmatrix} 0 & 7 \\ -1 & 1 \end{pmatrix} + (-1)^5(2)det\begin{pmatrix} 2 & 7 \\ 3 & 1 \end{pmatrix} + (-1)^6(-1)det\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}}$$
$$\boldsymbol{= (0) - (2)\big(2(1) - 3(7)\big) - \big(2(-1) - 3(0)\big) = -(2)(-19) - (-2) = 40}$$

[4] (20 points) (Similar to suggested exercise 4.4#5) Suppose that a matrix $A \in M_{n\times n}(F)$ can be written in the form $A = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}$ where $P, Q, 0, I$ are square matrices, 0 is the zero matrix, and I is the identity matrix. Prove that $det(A) = det(P)$. Justify the statements of your proof using information from Section 4.4 of the book.

Solution: Matrix A will have the form $\boldsymbol{A = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} \overbrace{\begin{matrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{matrix}}^{n} & \overbrace{\begin{matrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{matrix}}^{n} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}\right\} n \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{matrix} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}\right\} n \end{pmatrix}}$

Notice that matrix A is $2n \times 2n$. Evaluate $det(A)$ along the $2n^{th}$ row. There will be only one non-zero term:

$$\boldsymbol{det(A) = (-1)^{2n+2n}(1)det(\bar{A}_{(2n)(2n)}) = det\begin{pmatrix} \overbrace{\begin{matrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{matrix}}^{n} & \overbrace{\left.\begin{matrix} Q_{11} & \cdots & Q_{1(n-1)} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{n(n-1)} \end{matrix}\right\} n}^{n-1} \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \left.\begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{matrix}\right\} n-1 \end{pmatrix}}$$

Notice that in this expression the smaller matrix is $(2n-1)\times(2n-1)$. The big matrices on the diagonal are square, but not of the same size. The big matrices off the diagonal are not square. Evaluate the determinant along the $(2n-1)^{st}$ row. There will be only one non-zero term:

$$\boldsymbol{= (-1)^{(2n-1)+(2n-1)}(1)det\begin{pmatrix} \overbrace{\begin{matrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{matrix}}^{n} & \overbrace{\left.\begin{matrix} Q_{11} & \cdots & Q_{1(n-2)} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{n(n-2)} \end{matrix}\right\} n}^{n-2} \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \left.\begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{matrix}\right\} n-2 \end{pmatrix} =}$$

Continuing in this way, we will reach a point where we have the following expression:

$$\boldsymbol{= (-1)^{(n+2)+(n+2)}(1)det\begin{pmatrix} \overbrace{\begin{matrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{matrix}}^{n} & \left.\begin{matrix} Q_{11} \\ \vdots \\ Q_{n1} \end{matrix}\right\} n \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & 1 \end{pmatrix} =}$$

In this expression, the coefficient is (1) and the smaller matrix is $(n+1)\times(n+1)$. Evaluate its determinant along the $(n+1)^{st}$ row. There will be only one non-zero term:

$$\boldsymbol{= (-1)^{(n+1)+(n+1)}(1)det\begin{pmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{pmatrix} =}$$

In this expression, the coefficient is (1) and the smaller matrix is just matrix P. Stringing all the equations together, we have $det(A) = det(P)$.

[5] (20 points) (Similar to suggested exercise 5.1#2(b))
Let V be the vector space $V = P_1(R)$ with basis $\beta = \{v_1, v_2\} = \{3+2x, 4+3x\}$
Let $T: P_1(R) \to P_1(R)$ be the linear transform $T(a+bx) = (-11a+12b) + (-6a+6b)x$.

(a) Compute $[T]_\beta$. Show all details of the calculation and explain your steps.

Solution: We compute the outputs that result when the basis vectors are used as input.

$$\boldsymbol{T(v_1) = T(3+2x) = (-11(3)+12(2)) + (-6(3)+6(2))x = -9-6x = -3v_1}$$
$$\boldsymbol{T(v_2) = T(4+3x) = (-11(4)+12(3)) + (-6(4)+6(3))x = -8-6x = -2v_2}$$

These two calculations tell us that the matrix representation of T is $[T]_\beta = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$

(b) Are the basis vectors v_1, v_2 eigenvectors for T? Explain why.

Because $T(v_1) = constant \cdot v_1$, we conclude that v_1 is an eigenvector.
Because $T(v_2) = constant \cdot v_2$, we conclude that v_2 is an eigenvector.

(c) If the vectors v_1, v_2 are eigenvectors for T, then what are the eigenvalues? Explain.

Because $T(v_1) = -3v_1$, we conclude that v_1 has eigenvalue $\lambda_1 = -3$.
Because $T(v_2) = -2v_2$, we conclude that v_2 has eigenvalue $\lambda_1 = -2$.