Solutions for 2013-2014 Spring Semester MATH 3210/5210 (Barsamian) Homework 8
(Due Wednesday, April 9, 2014)

(Three problems similar to suggested exercise 5-1#3(a),(b),(c))
For each matrix $A \in M_{n\times n}(F)$

(i) Determine all the eigenvalues of A. Show all details of the calculation.
(ii) For each eigenvalue λ of A, find the eigenvectors corresponding to λ.
(iii) If possible, find a basis for F^n consisting of eigenvectors of A.
(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

[1] (20 points) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} \in M_{3\times 3}(R)$

(i) Determine all the eigenvalues of A. Show all details of the calculation.

Solution: We start by finding the characteristic polynomial $f(\lambda)$.

$$f(\lambda) = det(A - \lambda I_3) = det\begin{pmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{pmatrix} = do\ determinant\ along\ 3^{rd}\text{ column} =$$

$$= (-1)det\begin{pmatrix} 2 & (2-\lambda) \\ 2 & 2 \end{pmatrix} - (-1)det\begin{pmatrix} (3-\lambda) & 1 \\ 2 & 2 \end{pmatrix} + (-\lambda)det\begin{pmatrix} (3-\lambda) & 1 \\ 2 & (2-\lambda) \end{pmatrix}$$

$$= (-1)\big((2)(2)-(2)(2-\lambda)\big) + \big((3-\lambda)(2)-(2)(1)\big) - (\lambda)\big((3-\lambda)(2-\lambda)-(2)(1)\big)$$

$$= \cdots = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

(A check with Wolfram Alpha confirms that this is correct.)

The eigenvalues of A are the roots of the characteristic polynomial. We find them by setting $f(\lambda)$ equal to zero and solving for λ.

$$0 = f(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda-2)^2$$

We see that the eigenvalues will be $\lambda_1 = 1$ and $\lambda_2 = 2$. (Wolfram Alpha confirms this.)

(ii) For each eigenvalue λ of A, find the eigenvectors corresponding to λ.

Solution: Find the eigenvector corresponding to eigenvalue $\lambda_1 = 1$.

Corresponding to eigenvalue $\lambda_1 = 1$ will be an eigenvector that we can denote $v_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}$.

The vector v_1 must satisfy the matrix equation $(A - \lambda_1 I_3)v_1 = 0$. That is

$$\begin{pmatrix} 3-\lambda_1 & 1 & -1 \\ 2 & 2-\lambda_1 & -1 \\ 2 & 2 & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $(A - \lambda_1 I_3|0) = \left(\begin{array}{ccc|c} 3-\lambda_1 & 1 & -1 & 0 \\ 2 & 2-\lambda_1 & -1 & 0 \\ 2 & 2 & -\lambda_1 & 0 \end{array}\right)$

Using the value $\lambda_1 = 1$, this augmented matrix becomes $(A - \lambda_1 I_3|0) = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & 2 & -1 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Add (-1)R1 to R2.
- Add (-1)R1 to R3.
- Add R3 to R1.
- Multiply R1 by (1/2).
- Multiply R3 by (-1).
- Interchange R2 and R3.

We obtain the reduced row echelon form $\left(\begin{array}{ccc|c} 1 & 0 & (-1/2) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ (Wolfram Alpha confirms this.)

This corresponds to the system $\begin{cases} v_{11} - (1/2)v_{31} = 0 \\ v_{21} = 0 \\ 0 = 0 \end{cases}$

Clearly v_{21} must be zero. But v_{11} can be any real number, as long as $v_{31} = 2v_{11}$. So the solution set is $K_H = \{(s, 0, 2s), s \in R\}$. A basis for this solution set is $\beta = \{(1,0,2)\}$. We can let the basis vector be v_1. That is, we can let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

We check by multiplying $Av_1 = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3(1) + 1(0) - 1(2) \\ 2(1) + 2(0) - 1(2) \\ 2(1) + 2(0) + 0(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = v_1$

This confirms that v_1 is in fact an eigenvector with eigenvalue 1.

Now find the eigenvector corresponding to eigenvalue $\lambda_2 = 2$.

Corresponding to eigenvalue $\lambda_2 = 1$ will be an eigenvector that we can denote $v_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}$.

The vector x must satisfy the matrix equation $(A - \lambda_2 I_3)v_2 = 0$. That is

$$\begin{pmatrix} 3 - \lambda_2 & 1 & -1 \\ 2 & 2 - \lambda_2 & -1 \\ 2 & 2 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $(A - \lambda_2 I_3|0) = \left(\begin{array}{ccc|c} 3 - \lambda_2 & 1 & -1 & 0 \\ 2 & 2 - \lambda_2 & -1 & 0 \\ 2 & 2 & -\lambda_2 & 0 \end{array}\right)$

Using the value $\lambda_2 = 2$, this augmented matrix becomes $(A - \lambda_2 I_3|0) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Add (-2)R1 to R2.
- Add (-2)R1 to R3.
- Multiply R2 by (-1/2).
- Add (-2)R2 to R1.

We obtain the echelon form $\left(\begin{array}{ccc|c} 1 & 0 & (-1/2) & 0 \\ 0 & 1 & (-1/2) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ corresponding to system $\begin{cases} v_{12} - (1/2)v_{32} = 0 \\ v_{22} - (1/2)v_{32} = 0 \\ 0 = 0 \end{cases}$

v_{32} can be any real number, as long as $v_{12} = (1/2)v_{32}$ and $v_{22} = (1/2)v_{32}$. So the solution set is $K_H = \{((1/2)s, (1/2)s, s), s \in R\}$. A basis for this set is $\beta = \{(1,1,2)\}$.

We can let the basis vector be v_2. That is, we can let $v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

We check by multiplying $Av_1 = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3(1) + 1(1) - 1(2) \\ 2(1) + 2(1) - 1(2) \\ 2(1) + 2(1) + 0(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = 2v_2$

This confirms that v_2 is in fact an eigenvector with eigenvalue 2.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A. Solution: Since we only found two eigenvectors for A, we conclude that it is not possible to find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. Solution: Since it is not possible to find a basis for F^n consisting of eigenvectors of A, we conclude that A is not diagonalizable.

[2] (20 points) Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \in M_{2\times 2}(C)$

(i) Determine all the eigenvalues of A. Show all details of the calculation.

Solution: We start by finding the characteristic polynomial $f(\lambda)$.

$$f(\lambda) = det(A - \lambda I_2) = det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) - (2)(-2) = \lambda^2 - 2\lambda + 5$$

(Wolfram Alpha confirms this.)

The eigenvalues of A are the roots of the characteristic polynomial. We find them by setting $f(\lambda)$ equal to zero and solving for λ.

$$0 = f(\lambda) = \lambda^2 - 2\lambda + 5$$

It is not obvious how to factor this polynomial, so we use the quadratic formula.

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4\sqrt{-1}}{2} = \frac{2 \pm 4\mathrm{i}}{2} = 1 \pm 2\mathrm{i}$$

We see that the eigenvalues will be $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. (Wolfram Alpha confirms this.)

Remark: This tells us that the characteristic polynomial factors over the field C in the following way:

$$f(\lambda) = \lambda^2 - 2\lambda + 5 = (\lambda - \lambda_1)(\lambda - \lambda_2) = \big(\lambda - (1+2i)\big)\big(\lambda - (1-2i)\big)$$

(ii) For each eigenvalue $\boldsymbol{\lambda}$ of $\boldsymbol{A}$, find the eigenvectors corresponding to $\boldsymbol{\lambda}$.

Solution: Find the eigenvector corresponding to eigenvalue $\boldsymbol{\lambda_1 = 1 + 2i}$.

Corresponding to eigenvalue $\lambda_1 = 1 + 2i$ will be an eigenvector that we can denote $v_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$.

The vector v_1 must satisfy the matrix equation $(A - \lambda_1 I_2)v_1 = 0$. That is

$$\begin{pmatrix} 1-\lambda_1 & -2 \\ 2 & 1-\lambda_1 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $(A - \lambda_1 I_3 | 0) = \left(\begin{array}{cc|c} 1-\lambda_1 & -2 & 0 \\ 2 & 1-\lambda_1 & 0 \end{array}\right)$

Using the value $\lambda_1 = 1 + 2i$, this augmented matrix becomes

$$(A - \lambda_1 I_3 | 0) = \left(\begin{array}{cc|c} 1-(1+2i) & -2 & 0 \\ 2 & 1-(1+2i) & 0 \\ & & 0 \end{array}\right) = \left(\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right)$$

Perform the following row operations on this matrix:

- Multiply R1 by (i/2).
- Add (-2)R1 to R2.

We obtain the echelon form $\left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{11} - iv_{21} = 0 \\ 0 = 0 \end{cases}$

v_{21} can be any real number, as long as $v_{11} = iv_{21}$. So the solution set is $K_H = \{(is, s), s \in C\}$. A basis for this solution set is $\beta = \{(i, 1)\}$. We can let the basis vector be v_1. That is, we can let $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Check by multiplying $Av_1 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 1(i) - 2(1) \\ 2(i) + 1(1) \end{pmatrix} = \begin{pmatrix} -2+i \\ 1+2i \end{pmatrix} = (1+2i)\begin{pmatrix} i \\ 1 \end{pmatrix} = \lambda_1 v_1$

This confirms that $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_1 = 1 + 2i$.

Find the eigenvector corresponding to eigenvalue $\boldsymbol{\lambda_2 = 1 - 2i}$.

Corresponding to eigenvalue $\lambda_2 = 1 - 2i$ will be an eigenvector that we can denote $v_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$.

The vector v_2 must satisfy the matrix equation $(A - \lambda_2 I_2)v_1 = 0$. That is

$$\begin{pmatrix} 1-\lambda_2 & -2 \\ 2 & 1-\lambda_2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $(A - \lambda_2 I_3 | 0) = \left(\begin{array}{cc|c} 1-\lambda_2 & -2 & 0 \\ 2 & 1-\lambda_2 & 0 \end{array}\right)$

Using the value $\lambda_1 = 1 + 2i$, this augmented matrix becomes

$$(A - \lambda_2 I_3 | 0) = \left(\begin{array}{cc|c} 1-(1-2i) & -2 & 0 \\ 2 & 1-(1-2i) & 0 \\ & & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right)$$

Perform the following row operations on this matrix:

- Multiply R1 by (-i/2).
- Add (-2)R1 to R2.

We obtain the echelon form $\left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{12} + iv_{22} = 0 \\ 0 = 0 \end{cases}$

v_{22} can be any real number, as long as v_{12}=$-iv_{22}$. So the solution set is $K_H = \{(-is, s), s \in C\}$. A basis for this solution set is $\beta = \{(-i, 1)\}$. We can let the basis vector be v_2. That is, we can let $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Check by multiplying $Av_1 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1(-i) - 2(1) \\ 2(-i) + 1(1) \end{pmatrix} = \begin{pmatrix} -2 - i \\ 1 - 2i \end{pmatrix} = (1 - 2i)\begin{pmatrix} -i \\ 1 \end{pmatrix} = \lambda_2 v_2$

This confirms that $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_2 = 1 - 2i$.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

Solution: We found two eigenvectors, $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Because they are linearly independent vectors in C^2, we can use them as a basis of C^2. So our basis is $\beta = \{v_1, v_2\} = \left\{\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution: The invertible matrix Q can be made by using the eigenvectors $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as columns. That is $Q = (v_1 \quad v_2) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. We know that Q is invertible because it has rank 2.

The diagonal matrix D can be made by using the corresponding eigenvalues on the diagonal.

That is $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} (1 + 2i) & 0 \\ 0 & (1 - 2i) \end{pmatrix}$.

[3] (20 points) Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \in M_{2\times 2}(R)$

(i) Determine all the eigenvalues of A. Show all details of the calculation.

Solution: This is the same matrix that was used in problem [2], but now the field is R instead of C.
In problem [2], we found that the characteristic polynomial was $f(\lambda) = \lambda^2 - 2\lambda + 5$.
We used the quadratic formula to find roots of this polynomial, and the result was that we found two complex roots: $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Those are the eigenvalues when the field is C.
This told us that the characteristic polynomial factors over the field C in the following way:

$$f(\lambda) = \lambda^2 - 2\lambda + 5 = (\lambda - \lambda_1)(\lambda - \lambda_2) = \big(\lambda - (1 + 2i)\big)\big(\lambda - (1 - 2i)\big)$$

But now in that we are using the field R instead of C, we must say that **there are no real eigenvalues**!
Remark: This tells us that the polynomial $f(\lambda) = \lambda^2 - 2\lambda + 5$ does not factor over the field R.

(ii),(iii) Since there are no real eigenvalues, there are no eigenvectors, either.

(iv) Since there are no real eigenvalues, the matrix A is not diagonalizable over the field R.

[4] (20 points) Let $V = R^2$ and define $T: V \to V$ by $T(a, b) = (a + b, 4a + b)$.
Find the eigenvalues of T and find an ordered basis β for V such that $[T]_\beta$ is diagonal.

Solution:

First we get a matrix representation for T in the standard basis.

Let $\alpha = \{e_1, e_2\} = \{(1,0), (0,1)\}$ be the standard basis for R^2.

$[T]_\alpha$ will be a 2×2 matrix $[T]_\alpha = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{12} \end{pmatrix}$with entries obtained from the two equations:

$$\begin{cases} T(e_1) = A_{11}e_1 + A_{21}e_2 \\ T(e_2) = A_{12}e_1 + A_{22}e_2 \end{cases}$$

So we investigate what happens when these basis vectors are used as input to T.

$$\begin{cases} T(e_1) = T(1,0) = (1 + 0, 4(1) + 0) = (1,4) = 1(1,0) + 4(0,1) = (1)e_1 + (4)e_2 \\ T(e_2) = T(0,1) = (0 + 1, 4(0) + 1) = (1,1) = 1(1,0) + 1(0,1) = (1)e_1 + (1)e_2 \end{cases}$$

Matching the coefficients in the two sets of equations, we find $[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Now we find the characteristic polynomial and the eigenvalues of the matrix $[T]_\gamma$.

$$f(\lambda) = det([T]_\alpha - \lambda I_2) = det\left(\begin{pmatrix}1 & 1\\4 & 1\end{pmatrix} - \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}\right) = det\begin{pmatrix}(1-\lambda) & 1\\4 & (1-\lambda)\end{pmatrix}$$
$$= (1-\lambda)(1-\lambda) - (4)(1) = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

We conclude that the eigenvalues of the matrix $[T]_\alpha = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = 3$.

(Wolfram Alpha confirms this.)

Now we find the corresponding eigenvectors.

Find the eigenvector corresponding to eigenvalue $\lambda_1 = -1$.

Corresponding to eigenvalue $\lambda_1 = 1$ will be an eigenvector that we can denote $v_1 = \begin{pmatrix}v_{11}\\v_{21}\end{pmatrix}$.

The vector v_1 must satisfy the matrix equation $([T]_\alpha - \lambda_1 I_2)v_1 = 0$. That is,

$$\begin{pmatrix}(1-\lambda_1) & 1\\4 & (1-\lambda_1)\end{pmatrix}\begin{pmatrix}v_{11}\\v_{21}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

This corresponds to the augmented matrix $([T]_\alpha - \lambda_1 I_2|0) = \left(\begin{matrix}(1-\lambda_1) & 1\\4 & (1-\lambda_1)\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$

Using the value $\lambda_1 = -1$, this augmented matrix becomes $([T]_\alpha - \lambda_1 I_2|0) = \left(\begin{matrix}2 & 1\\4 & 2\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$

Perform the following row operations on this matrix:

- Add (-2)R1 to R2.
- Multiply R1 by (1/2).

We obtain the echelon form $\left(\begin{matrix}1 & (1/2)\\0 & 0\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$ corresponding to the system $\begin{cases}v_{11} + (1/2)v_{21} = 0\\ 0 = 0\end{cases}$

v_{11} can be any real number, as long as $v_{21} = -2v_{11}$. So the solution set is $K_H = \{(s,-2s), s \in R\}$, and a basis for the solution set is $\beta = \{(1,-2)\}$. We can let the basis vector be v_1. That is, let $v_1 = \begin{pmatrix}1\\-2\end{pmatrix}$.

Check by multiplying $[T]_\alpha v_1 = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1(1)+1(-2)\\4(1)+1(-2)\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix} = (-1)\begin{pmatrix}1\\-2\end{pmatrix} = \lambda_1 v_1$

This confirms that $v_1 = \begin{pmatrix}1\\-2\end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_1 = -1$.

Find the eigenvector corresponding to eigenvalue $\lambda_2 = 3$.

Corresponding to eigenvalue $\lambda_2 = 3$ will be an eigenvector that we can denote $v_2 = \begin{pmatrix}v_{12}\\v_{22}\end{pmatrix}$.

The vector v_2 must satisfy the matrix equation $([T]_\alpha - \lambda_2 I_2)v_2 = 0$. That is,

$$\begin{pmatrix}(1-\lambda_2) & 1\\4 & (1-\lambda_2)\end{pmatrix}\begin{pmatrix}v_{11}\\v_{21}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

This corresponds to the augmented matrix $([T]_\alpha - \lambda_2 I_2|0) = \left(\begin{matrix}(1-\lambda_2) & 1\\4 & (1-\lambda_2)\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$

Using the value $\lambda_2 = 3$, this augmented matrix becomes $([T]_\alpha - \lambda_2 I_2|0) = \left(\begin{matrix}-2 & 1\\4 & -2\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$

Perform the following row operations on this matrix:

- Add (2)R1 to R2.
- Multiply R1 by (-1/2).

We obtain the echelon form $\left(\begin{matrix}1 & (-1/2)\\0 & 0\end{matrix}\middle|\begin{matrix}0\\0\end{matrix}\right)$ corresponding to the system $\begin{cases}v_{12} + (-1/2)v_{22} = 0\\ 0 = 0\end{cases}$

v_{12} can be any real number, as long as $v_{22} = 2v_{12}$. So the solution set is $K_H = \{(s,2s), s \in R\}$, and a basis for the solution set is $\beta = \{(1,2)\}$. We can let the basis vector be v_2. That is, we can let $v_2 = \begin{pmatrix}1\\2\end{pmatrix}$.

Check by multiplying $[T]_\alpha v_1 = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1(1)+1(2)\\4(1)+1(2)\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} = (3)\begin{pmatrix}1\\2\end{pmatrix} = \lambda_2 v_2$

This confirms that $v_2 = \begin{pmatrix}1\\2\end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_2 = 3$.

Conclusion

Switching from column form to ordered-pair form for the eigenvectors, we have the following:

The vector $v_1 = (1,-2)$ is an eigenvector with eigenvalue $\lambda_1 = -1$.

The vector $v_2 = (1,2)$ is an eigenvector with eigenvalue $\lambda_2 = 3$.

Introduce ordered basis $\beta = \{v_1, v_2\} = \{(1,-2),(1,2)\}$.

In this basis, the matrix representation of T will be the diagonal matrix $[T]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

[5] (20 points) Let $V = P_2(R)$ and define $T: V \to V$ by $T\big(f(x)\big) = xf'(x) + f(2)x + f(3)$.

(a) Let $\alpha = \{v_1, v_2, v_3\} = \{1, x, x^2\}$, the standard basis for $P_2(R)$. Find $[T]_\alpha$.

Solution: Observe that the formula for T is given with a general function $f(x)$ as input. But we know that our inputs will be elements of $P_2(R)$, that is, functions of the form $f(x) = a + bx + cx^2$. So it will be helpful to have a formula for $T(a + bx + cx^2)$. For that, we get some of the parts first.

If $f(x) = a + bx + cx^2$, then

- The derivative of f will be $f'(x) = b + 2cx$.
- The value of f at $x = 2$ will be the number $f(2) = a + b(2) + c(2)^2 = a + 2b + 4c$.
- The value of f at $x = 3$ will be the number $f(3) = a + b(3) + c(3)^2 = a + 3b + 9c$.

We use those parts to get the formula for $T(a + bx + cx^2)$.

$$\begin{aligned} T(a + bx + cx^2) &= x(b + 2cx) + (a + 2b + 4c)x + (a + 3b + 9c) \\ &= (a + 3b + 9c) + (a + 3b + 4c)x + (2c)x^2 \end{aligned}$$

$[T]_\gamma$ will be a 3×3 matrix $[T]_\alpha = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{12} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ with entries obtained from the three equations:

$$\begin{cases} T(v_1) = A_{11}v_1 + A_{21}v_2 + A_{31}v_3 \\ T(v_2) = A_{12}v_1 + A_{22}v_2 + A_{32}v_3 \\ T(v_3) = A_{13}v_1 + A_{23}v_2 + A_{33}v_3 \end{cases}$$

So we investigate what happens when the basis vectors are used as input to T.

$$\begin{cases} T(v_1) = T(1 + 0x + 0x^2) = (1) + (1)x + (0)x^2 = (1)v_1 + (1)v_2 + (0)v_3 \\ T(v_2) = T(0 + 1x + 0x^2) = (3) + (3)x + (0)x^2 = (3)v_1 + (3)v_2 + (0)v_3 \\ T(v_3) = T(0 + 0x + 1x^2) = (9) + (4)x + (2)x^2 = (9)v_1 + (4)v_2 + (2)v_3 \end{cases}$$

Matching the coefficients in the two sets of equations, we find $[T]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

(b) Find the eigenvalues of the matrix $[T]_\alpha$ and the corresponding eigenvectors of $[T]_\alpha$. These eigenvectors should be column vectors representing elements of R^3.

Solution:

We start by finding the characteristic polynomial and the eigenvalues.

The characteristic polynomial is

$$f(\lambda) = det([T]_\alpha - \lambda I_3) = det\begin{pmatrix} (1-\lambda) & 3 & 9 \\ 1 & (3-\lambda) & 4 \\ 0 & 0 & (2-\lambda) \end{pmatrix} = (2-\lambda)det\begin{pmatrix} (1-\lambda) & 3 \\ 1 & (3-\lambda) \end{pmatrix}$$

$$= (2-\lambda)[(1-\lambda)(3-\lambda) - (1)(3)] = (2-\lambda)[\lambda^2 - 4\lambda] = (2-\lambda)(\lambda)(\lambda - 4)$$

We conclude that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$ and $\lambda_3 = 4$. (Wolfram Alpha confirms this.)

Now we find the corresponding eigenvectors.

Find the eigenvector corresponding to eigenvalue $\lambda_1 = 0$.

Corresponding to eigenvalue $\lambda_1 = 0$ will be an eigenvector that we can denote $v_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}$.

The vector v_1 must satisfy the matrix equation $([T]_\alpha - \lambda_1 I_3)v_1 = 0$. That is,

$$\begin{pmatrix} (1-\lambda_1) & 3 & 9 \\ 1 & (3-\lambda_1) & 4 \\ 0 & 0 & (2-\lambda_1) \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $([T]_\alpha - \lambda_1 I_3|0) = \left(\begin{array}{ccc|c} (1-\lambda_1) & 3 & 9 & 0 \\ 1 & (3-\lambda_1) & 4 & 0 \\ 0 & 0 & (2-\lambda_1) & 0 \end{array}\right)$

Using the value $\lambda_1 = 0$, this augmented matrix becomes $([T]_\alpha - \lambda_1 I_2|0) = \left(\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Add (-1)R1 to R2.
- Multiply R2 by (-1/5).
- Add (-9)R2 to R1.
- Add (-2)R2 to R3.

We obtain the echelon form $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{11} + 3v_{21} = 0 \\ v_{31} = 0 \\ 0 = 0 \end{cases}$

Clearly, we must have $v_{31} = 0$. We see that v_{21} can be any real number, as long as $v_{11}=-3v_{21}$.
So the solution set is $K_H = \{(-3s, s, 0), s \in R\}$, and a basis for this solution set is $\beta = \{(-3,1,0)\}$.

We can let the basis vector be v_1. That is, we can let $v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$.

Check by multiplying$[T]_\alpha v_1 = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \cdots = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (0)v_1 = \lambda_1 v_1$

This confirms that $v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_1 = 0$.

Find the eigenvector corresponding to eigenvalue $\lambda_2 = 2$.

Corresponding to eigenvalue $\lambda_2 = 2$ will be an eigenvector that we can denote $v_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}$.

The vector v_2 must satisfy the matrix equation $([T]_\alpha - \lambda_2 I_3)v_2 = 0$. That is,

$$\begin{pmatrix} (1-\lambda_2) & 3 & 9 \\ 1 & (3-\lambda_2) & 4 \\ 0 & 0 & (2-\lambda_2) \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $([T]_\alpha - \lambda_2 I_3|0) = \left(\begin{array}{ccc|c} (1-\lambda_2) & 3 & 9 & 0 \\ 1 & (3-\lambda_2) & 4 & 0 \\ 0 & 0 & (2-\lambda_2) & 0 \end{array}\right)$

Using the value $\lambda_2 = 2$, this augmented matrix becomes $([T]_\alpha - \lambda_2 I_2|0) = \left(\begin{array}{ccc|c} -1 & 3 & 9 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Add R1 to R2.
- Multiply R2 by (1/4).
- Add (-3)R2 to R1.
- Multiply R1 by (-1).

We obtain the echelon form $\left(\begin{array}{ccc|c} 1 & 0 & (3/4) & 0 \\ 0 & 1 & (13/4) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{12} + (3/4)v_{32} = 0 \\ v_{22} + (13/4)v_{32} = 0 \\ 0 = 0 \end{cases}$

v_{32} can be any real number, as long as $v_{12} = (-3/4)v_{32}$ and $v_{22} = (-13/4)v_{32}$. So the solution set is $K_H = \{((-3/4)s, (-13/4)s, s), s \in R\}$. A basis for the solution set is $\beta = \{(-3,-13,4)\}$.

We can let the basis vector be v_2. That is, we can let $v_2 = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$.

Check by multiplying $[T]_\alpha v_1 = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix} = \cdots = \begin{pmatrix} -6 \\ -26 \\ 8 \end{pmatrix} = (-2)v_2 = \lambda_2 v_2$

This confirms that $v_2 = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_2 = 2$.

Find the eigenvector corresponding to eigenvalue $\boldsymbol{\lambda_3 = 4}$.

Corresponding to eigenvalue $\lambda_3 = 4$ will be an eigenvector that we can denote $v_3 = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix}$.

The vector v_3 must satisfy the matrix equation $([T]_\alpha - \lambda_3 I_3)v_3 = 0$. That is,

$$\begin{pmatrix} (1-\lambda_3) & 3 & 9 \\ 1 & (3-\lambda_3) & 4 \\ 0 & 0 & (2-\lambda_3) \end{pmatrix}\begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to the augmented matrix $([T]_\alpha - \lambda_3 I_3|0) = \left(\begin{array}{ccc|c} (1-\lambda_3) & 3 & 9 & 0 \\ 1 & (3-\lambda_3) & 4 & 0 \\ 0 & 0 & (2-\lambda_3) & 0 \end{array}\right)$

Using the value $\lambda_3 = 4$, this augmented matrix becomes $([T]_\alpha - \lambda_3 I_2|0) = \left(\begin{array}{ccc|c} -3 & 3 & 9 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Multiply R1 by (-1/3).
- Add (-1)R1 to R2.
- Add (3)R3 to R2.
- Add (3)R2 to R1.
- Add (2)R2 to R3.

We obtain the echelon form $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{13} - v_{23} = 0 \\ v_{33} = 0 \\ 0 = 0 \end{cases}$

Clearly we must have $v_{33} = 0$. We see that v_{13} can be any real number, as long as $v_{23} = v_{13}$. So the solution set is $K_H = \{(s, s, 0), s \in R\}$. A basis for the solution set is $\beta = \{(1,1,0)\}$.

We can let the basis vector be v_3. That is, we can let $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Check by multiplying $[T]_\alpha v_1 = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \cdots = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} = (4)v_3 = \lambda_3 v_3$

This confirms that $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_3 = 4$.

(c) Using your result from (b), find a corresponding ordered basis $\boldsymbol{\beta}$ for$\boldsymbol{P_2(R)}$ such that $\boldsymbol{[T]_\beta}$ is a diagonal matrix. (The basis vectors in $\boldsymbol{\beta}$ should be polynomials, elements of $\boldsymbol{P_2(R)}$.)

Solution: Converting our column vectors to functions, we have

The eigenvector $v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ correspond to

the functions $v_1 = -3 + x$ and $v_2 = -3 - 13x + 4x^2$ and $v_3 = 1 + x$.

So our basis of eigenvectors is $\beta = \{v_1, v_2, v_3\} = \{-3 + x, \quad -3 - 13x + 4x^2, \quad 1 + x\}$.

In this basis, the matrix representation of T will be $[T]_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.