[1] (Five True False questions from suggested exercise 5.2#1.)

5.2#1(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable. **FALSE This is discussed in the middle of page 262. The identity matrix is used as an example: It has only one eigenvalue, but it is a diagonal matrix.**

5.2#1(c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T. **TRUE The definition of E_λ is that it is the set of eigenvectors of T that have eigenvalue λ. (page 264)**

5.2#1(d) If λ_1, λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$ **This statement is TRUE, but I can't find it stated explicitly in the chapter. It is a consequence of Theorem 5.5. To see why, suppose that there were some non-zero vector $v \in E_{\lambda_1} \cap E_{\lambda_2}$. Then we could define $v_1 = v$ and also define $v_2 = v$. Then v_1, v_2 are eigenvectors corresponding to the distinct eigenvalues λ_1, λ_2, and yet $\{v_1, v_2\}$ is not linearly independent. This contradicts Theorem 5.5.**

5.2#1(e) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \cdots, v_n\}$ be and ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose j^{th} column is v_j (for all $1 \le j \le n$), then $Q^{-1}AQ$ is a diagonal matrix. **TRUE This is discussed in Section 5.1 on page 251.**

5.2#1(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ. **TRUE This is statement (a) of Theorem 5.9.**

For problems [2],[3],[4], you are given a matrix $A \in M_{n\times n}(R)$.

(a) Test matrix A for diagonalizability.

(b) If A is diagonalizable, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

[2] Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in M_{2\times 2}(R)$.

Solution:

We find the eigenvalues of A.

The characteristic polynomial is $f(\lambda) = det\begin{pmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$. So the only eigenvalue is $\lambda = 1$; it has multiplicity 2.

Find the eigenvectors

We seek vectors $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that are solutions of the homogeneous system $\begin{pmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Using the eigenvalue $\lambda = 1$, this equation becomes $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. That is $\begin{cases} 0x_1 + 0x_2 = 0 \\ 2x_1 + 0x_2 = 0 \end{cases}$

Clearly, x_1 must be zero, but x_2 can be any real number.

So the solution set is the eigenspace $K_H = E_\lambda = E_1 = \{(0,s), s \in R\}$

A basis for this eigenspace is $\beta = \{(0,1)\}$. We can use the basis vector as an eigenvector. So $v_1 = (0,1)$ is an eigenvector with eigenvalue $\lambda = 1$.

Conclusion

Since A has only one eigenvector, it does not have a basis of eigenvectors. So A is not diagonalizable.

[3] Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \in M_{2\times 2}(R)$.

Solution: We find the eigenvalues of A.

Observe that the matrix $A - \lambda I_3 = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{pmatrix}$ is upper-triangular. By exercise 4.4#1(g) (from Homework 7), the determinant is the product of its diagonal entries. Therefore, the characteristic polynomial is $f(\lambda) = det\begin{pmatrix}1-\lambda & 1 & 0\\ 0 & 2-\lambda & 3\\ 0 & 0 & 2-\lambda\end{pmatrix} = (1-\lambda)(2-\lambda)^2$. So the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = 2$ with multiplicity 2. (Remark: Suggested problem 5.1#9 asks you to prove that the eigenvalues of an upper triangular matrix are just the diagonal entries of the matrix. This is an example.)

Find the eigenvectors

We seek vectors $v = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ that are solutions of $(A - \lambda I_3)v = \begin{pmatrix}1-\lambda & 1 & 0\\ 0 & 2-\lambda & 3\\ 0 & 0 & 2-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

This equation corresponds to the augmented matrix $(A - \lambda I_3|0) = \left(\begin{array}{ccc|c}1-\lambda & 1 & 0 & 0\\ 0 & 2-\lambda & 3 & 0\\ 0 & 0 & 2-\lambda & 0\end{array}\right)$

Find the eigenvectors corresponding to the eigenvalue $\boldsymbol{\lambda_1 = 1}$.

Using the eigenvalue $\lambda_1 = 1$, the augmented matrix $(A - \lambda I_3|0)$ becomes $\left(\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 1 & 0\end{array}\right)$.

Perform the following row operations on this matrix:

- Add (-1)R1 to R2.
- Add (-3)R3 to R2.
- Interchange R2 and R3.

These operations yield the form $\left(\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$, corresponding to the system $\begin{cases}0x_1 + 1x_2 + 0x_3 = 0\\ 0x_1 + 0x_2 + 1x_3 = 0\\ 0 = 0\end{cases}$

Clearly, x_1 can be any real number, but x_2 and x_3 must be zero.
So the solution set is the eigenspace $K_H = E_{\lambda_1} = E_1 = \{(1,0,0), s \in R\}$
A basis for this eigenspace is $\beta = \{(1,0,0)\}$. We can use the basis vector as an eigenvector.
So $v_1 = (1,0,0)$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

Check: $Av_1 = \begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 3\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1+0+0\\ 0+0+0\\ 0+0+0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = (1)v_1$. Good!

Find the eigenvectors corresponding to the eigenvalue $\boldsymbol{\lambda_2 = 2}$.

Using the eigenvalue $\lambda_2 = 2$, the augmented matrix $(A - \lambda I_3|0)$ becomes $\left(\begin{array}{ccc|c}-1 & 1 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$.

Perform the following row operations on this matrix:

- Multiply R1 by (-1)
- Multiply R2 by (1/3)

These operations yield the form $\left(\begin{array}{ccc|c}1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$, corresponding to the system $\begin{cases}x_1 - x_2 + 0x_3 = 0\\ 0x_1 + 0x_2 + x_3 = 0\\ 0 = 0\end{cases}$

Clearly x_1 can be any real number, as long as $x_2 = x_1$. And we must have $x_3 = 0$.
So the solution set is the eigenspace $K_H = E_{\lambda_2} = E_2 = \{(s, s, 0), s \in R\}$
A basis for this eigenspace is $\beta = \{(1,1,0)\}$. So $v_2 = (1,1,0)$ is an eigenvector with eigenvalue $\lambda_2 = 2$.

Check: $Av_2 = \begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 3\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}1+1+0\\ 0+2+0\\ 0+0+0\end{pmatrix} = \begin{pmatrix}2\\ 2\\ 0\end{pmatrix} = (2)v_2$. Good!

Conclusion

Since A has only two eigenvectors, it does not have a basis of eigenvectors. So A is not diagonalizable. In the fancy terminology of Section 5.2, we would say that the dimension of E_{λ_2} is 1, while the multiplicity of λ_2 is 2. Theorem 5.9 tells us that A is not diagonalizable.

[4] Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in M_{2\times 2}(R)$.

Solution: We find the characteristic polynomial and eigenvalues of A.

$$f(\lambda) = det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda + 3 - 8 = (\lambda+1)(\lambda-5)$$

So the eigenvalues are $\lambda_1 = -1$ with multiplicity 1 and $\lambda_2 = 5$ with multiplicity 1. By the corollary to Theorem 5.5 on page 261, we know that A will be diagonalizable because it has two eigenvalues.

Find the eigenvectors

We look for vectors $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that are solutions of $(A - \lambda I_2)v = \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

This equation corresponds to the augmented matrix $(A - \lambda I_2|0) = \left(\begin{array}{cc|c} 1-\lambda & 2 & 0 \\ 4 & 3-\lambda & 0 \end{array}\right)$

Find the eigenvectors corresponding to the eigenvalue $\lambda_1 = -1$.

Using the eigenvalue $\lambda_1 = -1$, the augmented matrix $(A - \lambda I_2|0)$ becomes $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right)$.

Perform the following row operations on this matrix:

- Multiply R1 by (-1)
- Add (-4)R1 to R2.

These operations yield the echelon form $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} x_1 + x_2 = 0 \\ 0 = 0 \end{cases}$

Clearly, x_1 can be any real number, as long $x_2 = -x_1$.

So the solution set is the eigenspace $K_H = E_{\lambda_1} = E_{-1} = \{(s, -s), s \in R\}$

A basis for this eigenspace is $\beta = \{(1, -1)\}$. So $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -1$.

Check: $Av_1 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)v_1$. Good!

Find the eigenvectors corresponding to the eigenvalue $\lambda_2 = 5$.

Using the eigenvalue $\lambda_2 = 5$, the augmented matrix $(A - \lambda I_2|0)$ becomes $\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right)$.

Perform the following row operations on this matrix:

- Add R1 to R2.
- Multiply R1 by (-1/4).

These operations yield the form $\left(\begin{array}{cc|c} 1 & (-1/2) & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} x_1 - (1/2)x_2 = 0 \\ 0 = 0 \end{cases}$

Clearly, x_2 can be any real number, as long as $x_1 = (1/2)x_2$. So the solution set is the eigenspace $K_H = E_{\lambda_2} = E_5 = \{(s, 2s), s \in R\}$

A basis for this eigenspace is $\beta = \{(1,2)\}$. So $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 5$.

Check: $Av_2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 4+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = (5)v_2$. Good!

Conclusion

By the discussion on page 251 of the book, we know that if we

- build a matrix Q using the two eigenvectors, that is if we let $Q = (v_2 \quad v_2) = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$
- build a diagonal matrix D, using the two eigenvalues, that is if we let $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

then the matrices A, D, Q will satisfy $Q^{-1}AQ = D$.

[5] Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in M_{2\times 2}(R)$. Find an expression for A^n, where n is an arbitrary integer.

Solution:

As discussed in class, if we know that a matrix is diagonalizable, then matrix powers are relatively easy. First, we get an expression for matrix A in terms of matrices Q and D.

$$\begin{aligned} Q^{-1}AQ &= D \\ QQ^{-1}AQ &= QD \\ AQ &= QD \\ AQQ^{-1} &= QDQ^{-1} \\ A &= QDQ^{-1} \end{aligned}$$

Now raise both sides to the n^{th} power.

$$A^n = (QDQ^{-1})^n = \overbrace{(QDQ^{-1})(QDQ^{-1})\cdots(QDQ^{-1})(QDQ^{-1})}^{n} = \overbrace{QDQ^{-1}QDQ^{-1}\cdots QDQ^{-1}QDQ^{-1}}^{n}$$
$$= Q\,\overbrace{DD\cdots DD}^{n}\,Q^{-1} = QD^nQ^{-1}$$

Now we use $Q = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ from problem [4].

Using Chapter 3 techniques, we find Q^{-1}.

Start with the augmented matrix $(Q|I_2)$ $\left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$.

Perform the following row operations on this matrix:

- Add R1 to R2.
- Multiply R2 by (1/3)
- Multiply R1 by (-1).
- Add R2 to R1.

These operations yield the form $\left(\begin{array}{cc|cc} 1 & 0 & (-2/3) & (1/3) \\ 0 & 1 & (1/3) & (1/3) \end{array}\right)$. This tells us that $Q^{-1} = \left(\frac{1}{3}\right)\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$.

We can check by multiplying: $\left(\frac{1}{3}\right)\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 2+1 & -2+2 \\ -1+1 & 1+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We can also use Wolfram Alpha to confirm Q^{-1}.

Find D^n.

Raising D to a power is easy because it is a diagonal matrix: $D^n = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}$.

Finally, compute A^n.

$$A^n = QD^nQ^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}\left(\left(\frac{1}{3}\right)\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\right) = \left(\frac{1}{3}\right)\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \left(\frac{1}{3}\right)\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} (-1)^n(-2) & (-1)^n \\ 5^n & 5^n \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 2(-1)^n + 5^n & (-1)(-1)^n + 5^n \\ (-2)(-1)^n + (2)5^n & (-1)^n + (2)5^n \end{pmatrix}$$

(Wolfram Alpha confirms this!)

Observe some special cases:

- With $n = 0$: $A^0 = \left(\frac{1}{3}\right)\begin{pmatrix} 2(-1)^0 + 5^0 & (-1)(-1)^0 + 5^0 \\ (-2)(-1)^0 + (2)5^0 & (-1)^0 + (2)5^0 \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- With $n = 1$: $A^1 = \left(\frac{1}{3}\right)\begin{pmatrix} 2(-1)^1 + 5^1 & (-1)(-1)^1 + 5^1 \\ (-2)(-1)^1 + (2)5^1 & (-1)^1 + (2)5^1 \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 3 & 6 \\ 12 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = A$
- With $n = 2$: $A^2 = \left(\frac{1}{3}\right)\begin{pmatrix} 2(-1)^2 + 5^2 & (-1)(-1)^2 + 5^2 \\ (-2)(-1)^2 + (2)5^2 & (-1)^2 + (2)5^2 \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 27 & 24 \\ 48 & 51 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$
- With $n = 3$: $A^3 = \left(\frac{1}{3}\right)\begin{pmatrix} 2(-1)^3 + 5^3 & (-1)(-1)^3 + 5^3 \\ (-2)(-1)^3 + (2)5^3 & (-1)^3 + (2)5^3 \end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix} 123 & 126 \\ 252 & 249 \end{pmatrix} = \begin{pmatrix} 41 & 42 \\ 84 & 83 \end{pmatrix}$

(Wolfram Alpha gives these same results for computing A^2 and A^3 directly.)