[1] (Five True False questions from suggested exercise 6.1#1.) The True/False answers for these questions are given in the back of the book. For each question, give the true/false answer and also explain why the answer is true or false. Justify your explanation by referring to things that the book says in Section 6.1.

6.1#1(a) An inner product is a scalar-valued function on the set of ordered pairs of vectors. **True. This is from the Definition of Inner Product at the bottom of page 329.**

6.1#1(c) An inner product is linear in both components. **False. As discussed in class, the inner product is linear in the first variable (part of the Definition of Inner Product at the bottom of page 329), but conjugate linear in the second variable (Theorem 6.1 on page 333).**

6.1#1(d) There is exactly one inner product on the vector space R^n. **False. See Example 2 on page 330.**

6.1#1(e) The triangle inequality only holds in finite-dimensional inner product spaces. **False. Theorem 6.2 tells us that the Triangle Inequality holds on any inner product space. There is no requirement that the space be finite-dimensional.**

6.1#1(h) If $\langle x, y\rangle = 0$ for all x in an inner product space, then $y = 0$. **True. Suppose that the equation $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \mathbf{0}$ is true for all $\boldsymbol{x}$ in an inner product space. Then in particular the equation is true if we let $\boldsymbol{x} = \boldsymbol{y}$. That is, $\langle \boldsymbol{y}, \boldsymbol{y}\rangle = \mathbf{0}$ is true. But by Theorem 6.1(d), $\langle \boldsymbol{y}, \boldsymbol{y}\rangle = \mathbf{0}$ can only be true if $\boldsymbol{y} = \mathbf{0}$.**

For problems [2],[3],[4], [5] do the following:

(a) Compute $\langle x, x\rangle$ and $\|x\|$.
(b) Compute $\langle y, y\rangle$ and $\|y\|$.
(c) Compute $\langle x, y\rangle$ and $|\langle x, y\rangle|$.
(d) Compute $\langle x + y, x + y\rangle$. (Hint: Use the results of (a),(b),(c)) and $\|x + y\|$.
(e) Verify both the Cauchy-Schwarz Inequality and the Triangle Inequality.
For all computations, show the exact answers in symbols involving fractions. If the exact answer is irrational, also give a decimal approximation rounded to three decimal places.

[2] Let $x = (2, -1,3) \in R^3$ and $y = (4,5, -7) \in R^3$. (Use the inner product defined in Example 1 on page 330.)
Solution:

(a) $\langle x, x\rangle = \langle (2, -1,3), (2, -1,3)\rangle = 2 \cdot 2 + (-1) \cdot (-1) + 3 \cdot 3 = 14.$

$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{14} \approx 3.741.$

(b) $\langle y, y\rangle = \langle (4,5, -7), (4,5, -7)\rangle = 4 \cdot 4 + 5 \cdot 5 + (-7) \cdot (-7) = 90.$

$\|y\| = \sqrt{\langle y, y\rangle} = \sqrt{90} = 3\sqrt{10} \approx 9.487.$

(c) $\langle x, y\rangle = \langle (2, -1,3), (4,5, -7)\rangle = 2 \cdot 4 + (-1) \cdot 5 + 3 \cdot (-7) = -18.$

$|\langle x, y\rangle| = |-18| = 18.$

(d) We can compute $\langle x + y, x + y\rangle$ by two methods.

Method #1: Compute it directly

$$\langle x + y, x + y\rangle = \langle (6,4, -4), (6,4, -4)\rangle = 6 \cdot 6 + 4 \cdot 4 + (-4) \cdot (-4) = 68$$

Method #2: Computing it by using inner product properties and previous results:

$\langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle = 14 + 2(-18) + 90 = 68$

$\|x + y\| = \sqrt{\langle x + y, x + y\rangle} = \sqrt{68} \approx 8.246$

(e) The Cauchy Schwarz inequality says $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$. Compare the left & right sides.

$Left\ side = |\langle x, y\rangle| = 18.$

$Right\ side = \|x\| \cdot \|y\| = \sqrt{14} \cdot \sqrt{90} = \sqrt{1260} \approx 35.496.$

Observe that $Left\ side \le Right\ side$ is true.

The Triangle inequality says $\|x + y\| \le \|x\| + \|y\|$. Compare the left & right sides.

$Left\ side = \|x + y\| = \sqrt{\langle x + y, x + y\rangle} = \sqrt{68} \approx 8.246.$

$Right\ side = \|x\| + \|y\| = \sqrt{14} + \sqrt{90} \approx 13.228.$

Observe that $Left\ side \leq Right\ side$ is true.

[3] Let $x = (2+i, 3-2i) \in C^2$ and $y = (-3+i, -1-2i) \in C^2$. (Use the inner product defined in Example 1 on page 330.)

Solution:

$(a)\langle x, x\rangle = \langle (2+i, 3-2i), (2+i, 3-2i)\rangle = (2+i)\cdot\overline{(2+i)} + (3-2i)\cdot\overline{(3-2i)}$
$= (2+i)\cdot(2-i) + (3-2i)\cdot(3+2i) = (2^2+1^2) + (3^2+2^2) = 4+1+9+4 = 18$

$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{18} \approx 4.243$

$(b)\langle y, y\rangle = \langle (-3+i, -1-2i), (-3+i, -1-2i)\rangle = (-3+i)\cdot\overline{(-3+i)} + (-1-2i)\cdot\overline{(-1-2i)}$
$= (-3+i)\cdot(-3-i) + (-1-2i)\cdot(-1+2i) = ((-3)^2+1^2) + ((-1)^2+(-2)^2)$
$= 9+1+1+4 = 15$

$\|y\| = \sqrt{\langle y, y\rangle} = \sqrt{15} \approx 3.873$

$(c)\langle x, y\rangle = \langle (2+i, 3-2i), (-3+i, -1-2i)\rangle = (2+i)\cdot\overline{(-3+i)} + (3-2i)\cdot\overline{(-1-2i)}$
$= (2+i)\cdot(-3-i) + (3-2i)\cdot(-1+2i) = (-6-5i+1) + (-3+8i+4)$
$= (-6+1-3+4) + (-5+8)i = -4+3i$

$|\langle x, y\rangle| = |-4+3i| = \sqrt{(-4)^2+3^2} = \sqrt{25} = 5$

(d) We could compute $\langle x+y, x+y\rangle$ by two methods, as we did in problem [1]. I will only present method #2.

Method #1: Compute it directly (You can do this at home, if you want.)

Method #2: Computing it by using inner product properties and previous results:

$$\langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} + \langle y, y\rangle$$
$$= 18 + (-4+3i) + \overline{(-4+3i)} + 15 = 18 + (-4+3i) + (-4-3i) + 15$$
$$= 18 - 8 + 15 = 25$$

$\|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{25} = 5$

(e) The Cauchy Schwarz inequality says $|\langle x, y\rangle| \leq \|x\|\cdot\|y\|$. Compare the left & right sides.

$Left\ side = |\langle x, y\rangle| = 5.$

$Right\ side = \|x\|\cdot\|y\| = \sqrt{18}\cdot\sqrt{15} = \sqrt{270} \approx 16.432.$

Observe that $Left\ side \leq Right\ side$ is true.

The Triangle inequality says $\|x+y\| \leq \|x\| + \|y\|$. Compare the left & right sides.

$Left\ side = \|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{25} = 5.$

$Right\ side = \|x\| + \|y\| = \sqrt{18} + \sqrt{15} \approx 8.116.$

Observe that $Left\ side \leq Right\ side$ is true.

[4] Let $x = A = \begin{pmatrix} 2 & 3+i \\ 1 & i \end{pmatrix} \in M_{2\times 2}(C)$ and $y = B = \begin{pmatrix} 4+i & 5i \\ 3 & 2 \end{pmatrix} \in M_{2\times 2}(C)$.

(i) Use the inner product defined in Example 4 on page 331.

(ii) Use the *Frobenius inner product*, defined in Example 5 on page 331.

Underlying Theory:

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(C)$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times 2}(C)$. We will compare the results of the two inner products $\langle A, B\rangle$ introduced on page 331

(i) Use the inner product defined in Example 4 on page 331.

(ii) Use the *Frobenius inner product*, defined in Example 5 on page 331.

(i) The inner product defined in Example 4 on page 331.

We identify matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(C)$ with the vector $A = (a, b, c, d) \in C^4$ and we identify matrix $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times 2}(C)$ with the vector $B = (e, f, g, h) \in C^4$.

Then we compute the inner product using the standard inner product for C^4.

$$\langle A,B\rangle = \langle(a,b,c,d),(e,f,g,h)\rangle = a\bar{e} + b\bar{f} + c\bar{g} + d\bar{h}$$

(ii) The *Frobenius inner product*, defined in Example 5 on page 331.

First, we find $B^* = \overline{\begin{pmatrix} e & f \\ g & h \end{pmatrix}}^t = \begin{pmatrix} \bar{e} & \bar{f} \\ \bar{g} & \bar{h} \end{pmatrix}^t = \begin{pmatrix} \bar{e} & \bar{g} \\ \bar{f} & \bar{h} \end{pmatrix}$.

Then compute $B^*A = \begin{pmatrix} \bar{e} & \bar{g} \\ \bar{f} & \bar{h} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{e}a + \bar{g}c & \bar{e}b + \bar{g}d \\ \bar{f}a + \bar{h}c & \bar{f}b + \bar{h}d \end{pmatrix}$.

Finally, $\langle A,B\rangle = tr(B^*A) = (\bar{e}a + \bar{g}c) + (\bar{f}b + \bar{h}d) = a\bar{e} + b\bar{f} + c\bar{g} + d\bar{h}$.

Observe that the two inner products give the same result! (We must be careful, though: We see that the two inner products give the same result for 2×2 matrices. We have not determined whether or not they give the same result for $n \times n$ matrices.) So we only need to use one of the methods to answer question [4]. I will use method (i), the inner product defined in Example 4.

Solution:

We identify matrix $A = \begin{pmatrix} 2 & 3+i \\ 1 & i \end{pmatrix} \in M_{2\times 2}(C)$ with vector $A = (2, 3+i, 1, i) \in C^4$ and identify matrix $B = \begin{pmatrix} 4+i & 5i \\ 3 & 2 \end{pmatrix} \in M_{2\times 2}(C)$ with vector $B = (4+i, 5i, 3, 2) \in C^4$.

$$(a)\langle x,x\rangle = \langle A,A\rangle = \langle(2,3+i,1,i),(2,3+i,1,i)\rangle = 2\cdot\bar{2} + (3+i)\cdot\overline{(3+i)} + 1\cdot\bar{1} + i\cdot\bar{i} =$$
$$= 2\cdot 2 + (3+i)\cdot(3-i) + 1\cdot 1 + i\cdot(-i) = 4 + 9 + 1 + 1 + 1 = 16$$
$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{16} = 4$$

$$(b)\langle y,y\rangle = \langle B,B\rangle = \langle(4+i,5i,3,2),(4+i,5i,3,2)\rangle = (4+i)\cdot\overline{(4+i)} + (5i)\cdot\overline{(5i)} + 3\cdot\bar{3} + 2\cdot\bar{2}$$
$$= (4+i)\cdot(4-i) + (5i)\cdot(-5i) + 3\cdot 3 + 2\cdot 2 = 16 + 1 + 25 + 9 + 4 = 55$$
$$\|y\| = \sqrt{\langle y,y\rangle} = \sqrt{55} \approx 7.416$$

$$(c)\langle x,y\rangle = \langle A,B\rangle = \langle(2,3+i,1,i),(4+i,5i,3,2)\rangle = 2\cdot\overline{(4+i)} + (3+i)\cdot\overline{(5i)} + 1\cdot\bar{3} + i\cdot\bar{2}$$
$$= 2\cdot(4-i) + (3+i)\cdot(-5i) + 1\cdot 3 + i\cdot 2 = 8 - 2i - 15i + 5 + 3 + 2i =$$
$$= (8+5+3) + (-2-15+2)i = 16 - 15i$$
$$|\langle x,y\rangle| = |16-15i| = \sqrt{16^2 + (-15)^2} = \sqrt{481} \approx 21.932$$

(d) We could compute $\langle x+y, x+y\rangle$ by two methods, as we did in problem [1]. I will only present method #2.

Method #1: Compute it directly (You can do this at home, if you want.)

Method #2: Computing it by using inner product properties and previous results:

$$\langle x+y,x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle x,y\rangle + \overline{\langle x,y\rangle} + \langle y,y\rangle$$
$$= 16 + (16-15i) + \overline{(16-15i)} + 55 = 16 + (16-15i) + (16+15i) + 55$$
$$= 16 + 16 + 16 + 55 = 103$$
$$\|x+y\| = \sqrt{\langle x+y,x+y\rangle} = \sqrt{103} \approx 10.149$$

(e) The Cauchy Schwarz inequality says $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$. Compare the left & right sides.

$Left\ side = |\langle x,y\rangle| = \sqrt{481} \approx 21.932$.

$Right\ side = \|x\|\cdot\|y\| = 4\cdot\sqrt{55} \approx 29.665$.

Observe that $Left\ side \le Right\ side$ is true.

The Triangle inequality says $\|x+y\| \le \|x\| + \|y\|$. Compare the left & right sides.

$$Left\ side = \|x+y\| = \sqrt{\langle x+y,x+y\rangle} = \sqrt{103} \approx 10.149$$
$$Right\ side = \|x\| + \|y\| = 4 + \sqrt{55} \approx 11.416$$

Observe that $Left\ side \le Right\ side$ is true.

[5] Let $x = f(t) = t^2 \in C([0,1])$ and $y = g(t) = e^t \in C([0,1])$. (Use the inner product defined in Example 3 on page 331.) (You are encouraged to use Wolfram Alpha to do the integrals, but your answer must include a clear presentation of both the integral and of the answer, with the answer in both symbolic form and decimal approximation. And be careful to type your integrals correctly!)

$$(a)\langle x,x\rangle = \langle f,f\rangle = \int_{t=0}^{t=1} f(t)\cdot f(t)dt = \int_{t=0}^{t=1} t^2\cdot t^2 dt = \int_{t=0}^{t=1} t^4 dt = \frac{t^5}{5}\bigg|_{t=0}^{t=1} = \frac{1}{5}$$

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{1/5} \approx 0.447$$

$$(b)\langle y,y\rangle = \langle g,g\rangle = \int_{t=0}^{t=1} g(t)\cdot g(t)dt = \int_{t=0}^{t=1} e^t\cdot e^t dt = \int_{t=0}^{t=1} e^{2t} t = \frac{e^{2t}}{2}\bigg|_{t=0}^{t=1} = \frac{e^2-e^0}{2} = \frac{e^2-1}{2} \approx 3.195$$

$$\|y\| = \sqrt{\langle y,y\rangle} = \sqrt{(e^2-1)/2} \approx 1.787$$

$$(c)\langle x,y\rangle = \langle f,g\rangle = \int_{t=0}^{t=1} f(t)\cdot g(t)dt = \int_{t=0}^{t=1} t^2\cdot e^t dt = e^t(t^2-2t+2)\big|_{t=0}^{t=1}$$

$$= \left(e^1(1^2-2(1)+2)\right) - \left(e^0(0^2-2(0)+2)\right) = e-2 \approx 0.718$$

$$|\langle x,y\rangle| = e-2 \approx 0.718$$

(d) We could compute $\langle x+y, x+y\rangle$ by two methods, as we did in problem [1]. But I will only present method #2.

Method #1: Compute it directly (You can do this at home, if you want.)

Method #2: Computing it by using inner product properties and previous results:

$$\langle x+y,x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle = \frac{1}{5} + 2(e-2) + \frac{e^2-1}{2}$$

$$= \frac{e^2}{2} + 2e - \frac{43}{10} \approx 4.831$$

$$\|x+y\| = \sqrt{\langle x+y,x+y\rangle} = \sqrt{\frac{e^2}{2} + 2e - \frac{43}{10}} \approx 2.198$$

(e) The Cauchy Schwarz inequality says $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$. Compare the left & right sides.

$$Left\ side = |\langle x,y\rangle| = e-2 \approx 0.718$$

$$Right\ side = \|x\|\cdot\|y\| = \sqrt{\frac{1}{5}}\cdot\sqrt{\frac{(e^2-1)}{2}} = \sqrt{\frac{(e^2-1)}{10}} \approx 0.799$$

Observe that $Left\ side \le Right\ side$ is true.

The Triangle inequality says $\|x+y\| \le \|x\| + \|y\|$. Compare the left & right sides.

$$Left\ side = \|x+y\| = \sqrt{\langle x+y,x+y\rangle} = \sqrt{\frac{e^2}{2} + 2e - \frac{43}{10}} \approx 2.198$$

$$Right\ side = \|x\| + \|y\| = \sqrt{\frac{1}{5}} + \sqrt{\frac{(e^2-1)}{2}} = 2.235$$

Observe that $Left\ side \le Right\ side$ is true.