[1] (30 points) (simplified version of assigned exercise 3.1#9)
Recall the elementary row operations:

Type (1) Interchange any two rows.
Type (2) Multiply any row by a nonzero scalar.
Type (3) Add any scalar multiple of one row to another row.

Consider the following elementary row operation of type (1): Interchange the 5^{th} and 7^{th} rows.
Find a sequence of four operations consisting of three Type (3) operations followed by one Type (2) operation that will accomplish the same result. Write your operations here:

Your first operation (Type (3)): ***Add Row5 to Row7***
Your second operation (Type (3)): ***Add (-1)Row7 to Row5***
Your third operation (Type (3)): ***Add Row5 to Row7***
Your fourth operation (Type (2)): ***Multiply Row5 by (-1)***

[2] (30 points) Simplified version of suggested exercise 3.3#4b

(a) Solve the system of linear equations $\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + 4x_3 = 2 \\ x_1 + 3x_2 + 6x_3 = 3 \end{cases}$

You may use the fact that $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & -3 & 1 \\ -8 & 5 & -2 \\ 3 & -2 & 1 \end{pmatrix}$

Solution:

The system corresponds to the matrix equation $Ax = b$ with $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 6 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Since we are given an inverse for matrix A, we know that A is invertible and we can use the inverse to find the solution. That is,

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ I_3 x &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

The value of x is $x = A^{-1}b = \begin{pmatrix} 6 & -3 & 1 \\ -8 & 5 & -2 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$

(b) **Check:**

$$Ax = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1(3) + 1(-4) + 1(2) \\ 2(3) + 3(-4) + 4(2) \\ 1(3) + 3(-4) + 6(2) \end{pmatrix} = \begin{pmatrix} 3 - 4 + 2 \\ 6 - 12 + 8 \\ 3 - 12 + 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = b$$

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[3] (30 points) (simplified version of suggested problem 3.4#2f)

Homogeneous system $\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 0 \\ x_2 + 2x_4 = 0 \end{cases}$ has solution set $K_H = \left\{ s\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, s \in R \right\}$

One solution to the non-homogeneous system $\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 5 \\ x_2 + 2x_4 = 3 \end{cases}$ is $v = \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix}$.

What is the solution set of the non-homogeneous system?

Solution:

Using the given information, $K = \{v\} + \{K_H\} = \left\{ v + s\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, s \in R \right\} = \left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, s \in R \right\}$.

[4] (40 points) (similar to sugg. exercise 3.3#4b) For the homogeneous system $\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$

(a) Find the solution set.
(b) Find a basis for the solution set.
(c) Check your answer to (b)
(d) What is the dimension of the solution set?

Solution: We need to solve the homogeneous system $\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$

The system corresponds to matrix equation $Ax = 0$, where $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It can also be represented by the single augmented matrix $(A|0) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$.

We can solve the homogeneous system by performing a sequence of elementary row operations on it to obtain a new system that has the same solution but that is easier to read. Alternately, we can do the same row operations on the corresponding augmented matrix. Here is a sequence that works.

- Add (-2)R1 to R2.
- Multiply R2 by(-1/3).
- Add (-2)R2 to R1

The result is the new augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$.

This augmented matrix corresponds to the system $\begin{cases} x_1 + 0 + x_3 = 0 \\ 0 + x_2 - x_3 = 0 \end{cases}$.

To satisfy this system, x_3 can be anything. But we must have $x_1 = -x_3$ and $x_2 = x_3$.

So the solution set is $K = \{(-s, s, s) \text{ } such \text{ } that \text{ } s \in R\}$.

(b) A basis for this solution set is $\boldsymbol{\beta} = \{(-1, 1, 1)\}$.

(c) We check this solution: $Ax = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1(-1) + 2(1) - 1(1) \\ 2(-1) + 1(1) + 1(1) \end{pmatrix} = \begin{pmatrix} -1 + 2 - 1 \\ -2 + 1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(d) Since the basis contains just one vector, the dimension of the solution set is $dim(K) = 1$.

[5] (40 points) (simplified version of 3.2#5) Given the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}$

(a) Find the rank of matrix A. Show all details of the calculation.
(b) If A is invertible, find its inverse.
(c) If you found an inverse for matrix A, check your work.

Solution: We form the augmented matrix $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$

We perform row operations on the corresponding augmented matrix. Here is a sequence that works.

- Add (-1)R1 to R2.
- Add (-1)R2 to R1.
- Add (-1)R2 to R3.
- Add R3 to R1.
- Add (-2)R3 to R2

The result is the new augmented matrix $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 1 \\ 0 & 1 & 0 & -3 & 3 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$.

This tells us that $M = \begin{pmatrix} 3 & -2 & 1 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ should be the inverse of matrix A.

We check:

$$\begin{aligned} AM &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 & 1 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1(3)+1(-3)+1(1) & 1(-2)+1(3)+1(-1) & 1(1)+1(-2)+1(1) \\ 1(3)+2(-3)+3(1) & 1(-2)+2(3)+3(-1) & 1(1)+2(-2)+3(1) \\ 0(3)+1(-3)+3(1) & 0(-2)+1(3)+3(-1) & 0(1)+1(-2)+3(1) \end{pmatrix} \\ &= \begin{pmatrix} 3-3+1 & -2+3-1 & 1-2+1 \\ 3-6+3 & -2+6-3 & 1-4+3 \\ 0-3+3 & 0+3-3 & 0-2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \end{aligned}$$

We conclude that our answer is correct. That is, M is the inverse of A. That is, $A^{-1} = \begin{pmatrix} 3 & -2 & 1 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix}$.

[6] (30 points) (simplified version of 3.4#4c)

A system of linear equations $\begin{cases} A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & + & A_{14}x_4 & = & b_1 \\ A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & + & A_{24}x_4 & = & b_2 \\ A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & + & A_{34}x_4 & = & b_3 \end{cases}$

corresponds to a matrix equation $Ax = b$ with $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

The augmented matrix is $(A|b) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & b_1 \\ A_{21} & A_{22} & A_{23} & A_{24} & b_2 \\ A_{31} & A_{32} & A_{33} & A_{34} & b_3 \end{pmatrix}$

Using Gaussian elimination, the augmented matrix $(A|b)$ can be put into reduced row echelon form $(A'|b')$.

Suppose that the reduced row echelon form is $(A'|b') = \begin{pmatrix} 1 & 1 & 0 & (-1/2) & 0 \\ 0 & 0 & 1 & (-1/2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Is the system consistent?

If the system is consistent, find all solutions.

If the system is not consistent, explain how you know that it is not consistent.

Solution: The solution is inconsistent. The reduced echelon form $(A'|b')$ corresponds to the system

$$\begin{cases} x_1 & + & x_2 & + & 0 & - & (1/2)x_4 & = & 0 \\ 0 & + & 0 & + & x_3 & - & (1/2)x_4 & = & 0 \\ 0 & + & 0 & + & 0 & + & 0 & = & 1 \end{cases}$$

There are no values of x_1, x_2, x_3, x_4 that will make the last equation $0 = 1$ true.