[1] (30 points) (part of suggested problem 5.1#2(c))
Define $T: R^3 \to R^3$ by $T(a,b,c) = \big((6a+4b-4c),(-8a-6b+4c),-2c\big)$.
Is $v = (0,1,1)$ an eigenvector of T? Explain.

Solution: Observe that

$$T(v) = T(0,1,1) = \Big(\big(6(0)+4(1)-4(1)\big),(-8(0)-6(1)+4c),-2(1)\Big) = (0,-2,-2)$$
$$= -2(0,1,1) = -2v$$

Conclude that v is an eigenvector with eigenvalue $\lambda = -2$.

[2] (30 points) (part of suggested problem 5.1#3(c))
Find the eigenvalues of $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \in M_{2\times 2}(C)$. Show all details of the calculation.

Solution: Find the characteristic polynomial

$$f(\lambda) = det\begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = (i-\lambda)(-i-\lambda) - 2(1) = (i)(-i) - i\lambda + i\lambda + \lambda^2 - 2 = 1 + \lambda^2 - 2$$
$$= \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

Conclude that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$.

[3] (30 points) (suggested problem 5.1#3(a)) The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in M_{2\times 2}(R)$ is diagonalizable.

(a) Determine all the eigenvalues of A.
(b) For each eigenvalue λ of A, find the eigenvectors corresponding to λ.
(c) Determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.
(Hint: Notice that you are not asked to find Q^{-1}. You don't need it to find D.)

(a) Determine all the eigenvalues of A.

Solution: The Characteristic Polynomial is $f(\lambda) = det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 =$ $(\lambda+1)(\lambda-4)$. Conclude that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

(b) For each eigenvalue λ of A, find the eigenvectors corresponding to λ.

Find the eigenvector corresponding to eigenvalue $\lambda_1 = -1$.

Solution: Corresponding to eigenvalue $\lambda_1 = -1$ will be an eigenvector that we can denote $v_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$.

The vector v_1 must satisfy the equation $(A - \lambda_1 I_2)v_1 = 0$. That is, $\begin{pmatrix} 1-\lambda_1 & 2 \\ 3 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This corresponds to the augmented matrix $(A - \lambda_1 I_3|0) = \left(\begin{array}{cc|c} 1-\lambda_1 & 2 & 0 \\ 3 & 2-\lambda_1 & 0 \end{array}\right)$

Using the value $\lambda_1 = -1$, this becomes $(A - \lambda_1 I_2|0) = \left(\begin{array}{cc|c} 1-(-1) & 2 & 0 \\ 3 & 2-(-1) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Multiply R1 by (1/2).
- Add (-3)R1 to R2.

We obtain the echelon form $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{11} + v_{21} = 0 \\ 0 = 0 \end{cases}$

We see that v_{11} can be any real number, as long as $v_{21} = -v_{11}$.

So the solution set is $K_H = \left\{\begin{pmatrix} s \\ -s \end{pmatrix}, s \in R\right\}$. A basis for this solution set is $\beta = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

We can let the eigenvector v_1 be the basis vector. That is, we let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Check by multiplying $Av_1 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1(1)+2(-1) \\ 3(1)+2(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 v_1$

This confirms that $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_1 = -1$.

Find the eigenvector corresponding to eigenvalue $\lambda_2 = 4$.

Solution: Corresponding to eigenvalue $\lambda_2 = 4$ will be an eigenvector that we can denote $v_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$.

The vector v_2 must satisfy the equation $(A - \lambda_1 I_2)v_2 = 0$. That is, $\begin{pmatrix} 1-\lambda_2 & 2 \\ 3 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

This corresponds to the augmented matrix $(A - \lambda_2 I_3|0) = \left(\begin{array}{cc|c} 1-\lambda_2 & 2 & 0 \\ 3 & 2-\lambda_2 & 0 \end{array}\right)$

Using the value $\lambda_2 = 4$, this becomes $(A - \lambda_2 I_2|0) = \left(\begin{array}{cc|c} 1-(4) & 2 & 0 \\ 3 & 2-(4) & 0 \end{array}\right) = \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right)$

Perform the following row operations on this matrix:

- Add R1 to R2.
- Multiply R1 by (-1/3)

We obtain the form $\left(\begin{array}{cc|c} 1 & (-2/3) & 0 \\ 0 & 0 & 0 \end{array}\right)$ corresponding to the system $\begin{cases} v_{12} + (-2/3)v_{22} = 0 \\ 0 = 0 \end{cases}$

We see that v_{22} can be any real number, as long as $v_{12} = (2/3)v_{22}$.

So the solution set is $K_H = \left\{\begin{pmatrix} (2/3)s \\ s \end{pmatrix}, s \in R\right\}$. A basis for this solution set is $\beta = \left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$.

We can let the eigenvector v_2 be the basis vector. That is, we let $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Check by multiplying $Av_1 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1(2)+2(3) \\ 3(2)+2(3) \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = (4)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \lambda_2 v_2$

This confirms that $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is in fact an eigenvector with eigenvalue $\lambda_2 = 4$.

(c) Determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.
(Hint: Notice that you are not asked to find Q^{-1}. You don't need it to find D.)

Solution: The invertible matrix Q can be made by using the eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as columns. That is $Q = (v_1 \quad v_2) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. We know that Q is invertible because it has rank 2.

The diagonal matrix D can be made by using the corresponding eigenvalues on the diagonal.

That is $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$.

[4] (30 points) Let $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{pmatrix}$

(a) What are the eigenvalues of A? Explain and/or show calculations.

Solution: We know that the eigenvalues of an upper-triangular matrix are just the diagonal entries. So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.

(b) Is A diagonalizable? Explain and/or show calculations.

Remark: Before presenting the correct solution, I want to comment on an incorrect solution that many of you submitted on your exam papers. Many of you answered that A is not diagonalizable, because it has only two eigenvalues. While it is true that A is not diagonalizable, the reason is not because it has only two eigenvalues. Indeed, observe that the matrix $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$, the same two eigenvalues as matrix A. And observe that matrix B is diagonalizable. (It is already a diagonal matrix!)

Correct Solution: The issue is whether or not there is a basis $\beta = \{v_1, v_2, v_3\}$ of R^3 consisting of vectors that are eigenvectors for A. If there is such a basis, then we know that we could use the three basis vectors to build an invertible matrix $Q = (v_1 \quad v_2 \quad v_3)$, and then use Q to find a diagonal matrix

$D = Q^{-1}AQ$. That is, we would know that matrix A is diagonalizable. So we have to determine the number of eigenvectors of A.

We know that there will be one eigenvector v_1 with eigenvalue $\lambda_1 = 3$. (Each eigenvalue has at least one associated eigenvector. But the number of associated eigenvectors cannot be greater than the multiplicity of the eigenvalue. Hence there will be exactly one eigenvector with eigenvalue $\lambda_1 = 3$.)

We know that there will be at least one eigenvector with eigenvalue $\lambda_2 = 5$. The question is whether there will be exactly one such eigenvector, or two.

We consider the rank and nullity of the matrix $(A - \lambda_2 I_3) = \begin{pmatrix} 3-\lambda_2 & 2 & 1 \\ 0 & 5-\lambda_2 & 7 \\ 0 & 0 & 5-\lambda_2 \end{pmatrix}$

Using the value $\lambda_2 = 5$, this matrix becomes $(A - \lambda_2 I_3) = \begin{pmatrix} -2 & 2 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$. This matrix has rank 2.

Therefore, by the dimension theorem, it has $nullity(A - \lambda_2 I_3) = 3 - 2 = 1$. Recall that the nullity is the number that is the dimension of the null space. The null space is the solution set of the matrix equation $(A - \lambda_2 I_3)v = 0$. That is, the equation $\begin{pmatrix} -2 & 2 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This solution set is also called the eigenspace, E_{λ_2}.

The fact that this eigenspace is one-dimensional tells us that it is spanned by a single basis vector. That in turn tells us that there is only one eigenvector associated to the eigenvalue $\lambda_2 = 5$.Since matrix A has only two eigenvectors, not three, we conclude that A is not diagonalizable.

(c) Is A invertible? Explain and/or show calculations.

Solution: Because A is upper-triangular, its determinant is just the product of its diagonal entries. (This is a fact from Chapter 4. So this is one place where Chapter 4 showed up on the exam.) So $det(A) = 3 \cdot 5 \cdot 5 = 75$. Since $det(A) \neq 0$, we know that A is invertible. (This is another fact from Chapter 4.)

[5] (30 points) Let $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

(a) What are the eigenvalues of A? Explain and/or show calculations.

Solution: As in problem [4], the eigenvalues of an upper-triangular matrix are just the diagonal entries. So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$ and $\lambda_3 = 0$.

(b) Is A diagonalizable? Explain and/or show calculations.

Solution: We know that each eigenvalue has exactly one eigenvector. (Each eigenvalue has at least one eigenvector, and the number of eigenvectors for each eigenvalue cannot exceed the multiplicity of the eigenvalue.) And we know that the eigenvectors associated to different eigenvalues are linearly independent of each other (By Theorem 5.5 on page 261). So A has three linearly independent eigenvectors. But a set of three linearly independent vectors in R^3 qualifies as a basis. Since there is a basis for R^3 consisting of eigenvectors of A, we conclude that A is diagonalizable. (Since there is a basis $\beta = \{v_1, v_2, v_3\}$ of R^3 consisting of eigenvectors for A, we could use the three basis vectors to build an invertible matrix $Q = (v_1 \quad v_2 \quad v_3)$, and then use Q to find a diagonal matrix $D = Q^{-1}AQ$.)

(c) Is A invertible? Explain and/or show calculations.

Solution: A is upper-triangular, so its determinant is the product of its diagonal entries: $det(A) = 3 \cdot 5 \cdot 0 = 0$. Since $det(A) = 0$, we know that A is not invertible.

[6] (70 points) An unknown matrix $A \in M_{2\times 2}(R)$ has

- eigenvector $v_1 = \begin{pmatrix}1\\2\end{pmatrix}$ with eigenvalue $\lambda_1 = 2$.
- eigenvector $v_2 = \begin{pmatrix}2\\3\end{pmatrix}$ with eigenvalue $\lambda_2 = 3$.

(a) Find the matrix Q and the diagonal matrix D such that $Q^{-1}AQ = D$.
(Hint: You have not yet been asked to find Q^{-1}. You don't need it to find D.)

Solution: We can build Q using the eigenvectors as columns: $Q = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}$.

And we can build D using the eigenvalues as diagonal entries: $D = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$.

(b) Find Q^{-1}.

Solution: We build the augmented matrix $(Q|I_2) = \left(\begin{array}{cc|cc}1 & 2 & 1 & 0\\2 & 3 & 0 & 1\end{array}\right)$

Perform the following row operations on this matrix:

- Add (-2)R1 to R2.
- Add (2)R2 to R1.
- Multiply R2 by (-1)

We obtain the new augmented matrix $\left(\begin{array}{cc|cc}1 & 0 & -3 & 2\\0 & 1 & 2 & -1\end{array}\right)$. This tells us that $\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix}$ should be the inverse of Q. But we should check by multiplying:

$$\begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix} = \begin{pmatrix}1(-3)+2(2) & 1(2)+2(-1)\\2(-3)+3(2) & 2(2)+3(-1)\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Since the product is the identity matrix, we conclude that $Q^{-1} = \begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix}$.

(c) Solve the expression $Q^{-1}AQ = D$ for matrix A in terms of matrices D, Q, Q^{-1}.

Solution:

$$\begin{aligned} Q^{-1}AQ &= D\\ QQ^{-1}AQ &= QD\\ I_2AQ &= QD\\ AQ &= QD\\ AQQ^{-1} &= QDQ^{-1}\\ AI_2 &= QDQ^{-1}\\ A &= QDQ^{-1}\end{aligned}$$

(d) Find a formula for matrix A^n, where n is an arbitrary positive integer.

Solution: $A^n = (QDQ^{-1})^n = \overbrace{(QDQ^{-1})(QDQ^{-1})\dots(QDQ^{-1})}^{n} = QDQ^{-1}QDQ^{-1}\dots QDQ^{-1}$

$$= QDI_2DI_2\dots I_2DQ^{-1} = Q\,\overbrace{DD\dots D}^{n}\,Q^{-1} = QD^nQ^{-1} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}^n\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}2^n & 0\\0 & 3^n\end{pmatrix}\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix} = \dots = \begin{pmatrix}(-3)2^n+(4)3^n & (2)2^n-(2)3^n\\(-6)2^n+(2)3^n & (4)2^n-(3)3^n\end{pmatrix}$$

(e) Using your answer from part (c) Find A.

Solution: $A = QDQ^{-1} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix} = \dots = \begin{pmatrix}6 & -2\\6 & -1\end{pmatrix}$

(f) Using your formula from part (d), find A^4.

Solution: $A^4 = QDQ^{-1} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}^4\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}2^4 & 0\\0 & 3^4\end{pmatrix}\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix}$

$$= \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}16 & 0\\0 & 81\end{pmatrix}\begin{pmatrix}-3 & 2\\2 & -1\end{pmatrix} = \dots = \begin{pmatrix}276 & -130\\390 & -179\end{pmatrix}$$

.