

Section 1: Image

Suppose that f is a function $f: A \rightarrow B$

If $x \in A$, then the symbol $f(x)$ denotes the output that results when x is used as input.

Notice that $f(x) \in B$.

Another name for the output $f(x)$ is “the image of x under the map f ”.

So the input x is an element of the domain A , while the image $f(x)$ is an element of the codomain B .

Examples

Example #1: For $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, we have the following:

- The image of 2 under the map f is 4, because $f(2) = 2^2 = 4$.
- The image of -2 under the map f is 4, because $f(-2) = (-2)^2 = 4$.

Example #2: For $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$, we have the following:

- The image of 2 under the map g is 8, because $g(2) = 2^3 = 8$.
- The image of -2 under the map g is -8 , because $g(-2) = (-2)^3 = -8$.

Section 2: Preimage

Suppose that f is a function $f: A \rightarrow B$

If $y \in B$, then the symbol $f^{-1}(y)$ denotes the set of all inputs that will yield y as an output.

Notice that $f^{-1}(y) \subset A$.

Another name for the set $f^{-1}(y) \subset A$ is “the preimage of y under the map f ”.

So the output y is an element of the codomain B , while the preimage $f^{-1}(y)$ is a subset of the domain A .

Examples:

Example #1: For $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, we have the following:

- The preimage of 4 under the map f is the set $\{-2, 2\}$, because $f(2) = 4$ and $f(-2) = 4$
- The preimage of 0 under the map f is the set $\{0\}$, because only $f(0) = 0$.
- The preimage of -5 under the map f is the the empty set ϕ , because there is no x such that $x^2 = -5$

Using the notation for the preimage, we would write

- $f^{-1}(4) = \{-2, 2\}$.
- $f^{-1}(0) = \{0\}$.
- $f^{-1}(-5) = \phi$.

Again notice that in each case, the preimage is a subset of the domain.

Example #2: For $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$, we have the following:

- The preimage of 8 under the map g is the set $\{2\}$, because only $g(2) = 8$.
- The preimage of 0 under the map g is the set $\{0\}$, because only $g(0) = 0$.

- The preimage of -5 under map g is the set $\{-(5)^{1/3}\}$, because $g(-(5)^{1/3}) = (-(5)^{1/3})^3 = -5$

Using the notation for the preimage, we would write

- $g^{-1}(8) = \{2\}$.
- $g^{-1}(0) = \{0\}$.
- $g^{-1}(-5) = \{-(5)^{1/3}\}$.

Again notice that in each case, the preimage is a subset of the domain.

Section 3: Inverse Functions

First, a definition

Definition of inverse functions:

words: f and g are inverse functions.

meaning: f and g are functions satisfy the following four conditions

- $f: A \rightarrow B$ for some domain A and some codomain B .
- $g: B \rightarrow A$. That is, the domain and codomain of g are the reverse of what they are for f .
- For all $x \in A$, the equation $g(f(x)) = x$ is true.
- For all $y \in B$, the equation $f(g(y)) = y$ is true.

The last two conditions are called the *inverse relations*.

Additional terminology: If f and g are inverse functions, we also say that g is an inverse function for f , and we also say that f is an inverse function for g .

Examples

Example #1: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^3$, and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be $g(x) = x^{1/3}$.

Observe that $g(f(x)) = (x^3)^{1/3} = x$.

Also observe that $f(g(y)) = ((y)^{1/3})^3 = y$.

So the inverse relations are both true. Conclude that f and g are inverse functions.

Example #2: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Then f does not have an inverse function. To see why, consider what happens when we try to come up with one. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = \sqrt{x}$.

- Observe that $g(f(2)) = \sqrt{2^2} = 2$, which is fine, but $g(f(-2)) = \sqrt{(-2)^2} = 2$. So the equation $g(f(x)) = x$ is not always true.
- Also observe that $f(g(5)) = (\sqrt{5})^2 = 5$ which is fine, but $f(g(-5)) = (\sqrt{-5})^2$ which does not even exist. So the equation $f(g(y)) = y$ is not always true.

Facts about inverse functions:

- A function $f: A \rightarrow B$ has an inverse function $g: B \rightarrow A$ if and only if f is one-to-one and onto.
- The inverse function g will also be one-to-one and onto.
- If some functions $f: A \rightarrow B$ and $g: B \rightarrow A$ satisfy both inverse relations:
 - For all $x \in A$, the equation $g(f(x)) = x$ is true.

- For all $y \in B$, the equation $f(g(y)) = y$ is true.
then it can be proven that f and g are both one-to-one and onto, so they qualify to be called inverse functions.
- Inverse functions are unique: A function can have only one inverse function.

Additional notation: If function $f: A \rightarrow B$ has an inverse function, we use the symbol f^{-1} to denote the unique inverse function.

Example:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3$. Then f has an inverse function $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f^{-1}(y) = y^{1/3}$.

Section 4: Using inverse notation:

Observe that inverse notation and preimage notation look the same. This is confusing. I will discuss some examples:

Example #1

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3$. What does the symbol $f^{-1}(8)$ mean? There are two possibilities.

- Remember that in Section 2 above, the symbol $f^{-1}(8)$ meant the preimage of 8 under the map f . The preimage is always a set, a subset of the domain. We wrote $f^{-1}(8) = \{2\}$. Notice that this is a set.
- But in Section 3, the symbol f^{-1} was used to denote the inverse function $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula $f^{-1}(y) = y^{1/3}$. In that usage, the symbol $f^{-1}(8)$ would denote the output that results when we feed the number y into the function f^{-1} . That is $f^{-1}(8) = (8)^{1/3} = 2$. Notice that this is a number, not a set.

Which interpretation of the meaning of the symbol $f^{-1}(8)$ is correct? Generally, if a function f has an inverse function, then we interpret the symbol $f^{-1}(8)$ to mean the number that results when an input of 8 is fed into the inverse function.. That is, we interpret the symbol $f^{-1}(8)$ to mean a single element of the domain.

Example #2

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. What does the symbol $f^{-1}(4)$ mean? There is only one possibility.

- As in Section 2 above, the symbol $f^{-1}(4)$ means the preimage of 4 under the map f . The preimage is always a set, a subset of the domain. In this case, it is $f^{-1}(4) = \{2, -2\}$.
- Because $f(x) = x^2$ has no inverse function, the symbol $f^{-1}(4)$ cannot be interpreted in terms of an inverse function. It can only be interpreted as a symbol for a preimage.

Section 5: What does all this have to Homework 5?

In problem [1], you are asked to find the image of an element of the domain. Based on the definitions above, you should now understand that in you are being asked to find $f(\vec{v}_1)$ in part (a) and to find $f(\vec{v}_2)$ in part (b).

In problem [4], you are again asked to find the image of an element of the domain. Based on the definitions above, you should now understand that in you are being asked to find $Rep_\beta(\vec{v}_1)$ in part (a) and to find $Rep_\beta(\vec{v}_2)$ in part (b).

In problem [5], you are asked to find $Rep_\beta^{-1}\left(\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}\right)$.

Based on the discussion above, we recognize that this either a question about preimage or a question about an inverse function. Which is it? Well, we know that the function $Rep_\beta: \mathcal{P}_2 \rightarrow \mathbb{R}^3$ is an isomorphism. So it is one-to-one and onto. That tells us that it has an inverse function, denoted $Rep_\beta^{-1}: \mathbb{R}^3 \rightarrow \mathcal{P}_2$. From the discussion above, we know that the inverse function is also one-to-one and onto. I mentioned in class (and it is also discussed in the book) that the inverse function also preserves vector space operations. In other words, the inverse function is also an isomorphism.

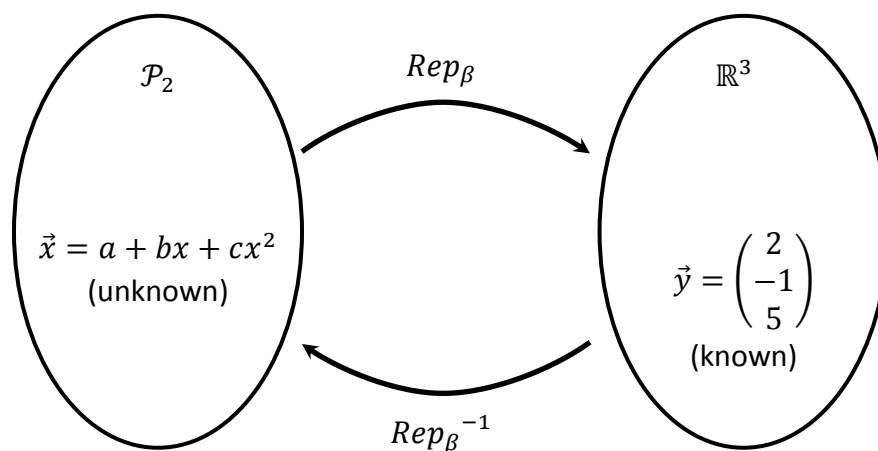
So how are you supposed to answer question [5]?

Note that vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ is an element of \mathbb{R}^3 . Give the vector the name \vec{y} , so $\vec{y} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \in \mathbb{R}^3$.

The symbol $Rep_\beta^{-1}(\vec{y})$ denotes the output that results from feeding the vector \vec{y} into the function Rep_β^{-1} . Note that this symbol $Rep_\beta^{-1}(\vec{y})$ represents an element of \mathcal{P}_2 . Give it the name \vec{x} , so $\vec{x} \in \mathcal{P}_2$. That is, the symbol \vec{x} denotes some unknown 2nd degree polynomial, so it represents something of the form $\vec{x} = a + bx + cx^2$.

You are trying to find the unknown vector \vec{x} such that $Rep_\beta^{-1}(\vec{y}) = Rep_\beta^{-1}\left(\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}\right) = \vec{x}$.

Consider the diagram below.



You are trying to find the vector $\vec{x} = a + bx + cx^2 \in \mathcal{P}_2$ that will make this diagram work.

That is, it is you are just trying to find the unknown vector \vec{x} such that $Rep_\beta(\vec{x}) = \vec{y} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.