[1] Use Gauss's method on each system. If there are no solutions, say "no solution" and explain why. If there is exactly one solution find it and check your solution by substitution. (Show the check clearly.) If there are many solutions, write "many solutions" and explain why.

(a) $\begin{cases} 2x+3y=3 \\ x-y=4 \end{cases}$ ➔ -2R₂+R₁ ➔ $\begin{cases} 5y=-5 \\ x-y=4 \end{cases}$ ➔ (1/5)R₁ ➔ $\begin{cases} y=-1 \\ x-y=4 \end{cases}$ ➔ R1 ⇔ R2 ➔ $\begin{cases} x-y=4 \\ y=-1 \end{cases}$

Second equation tells us $y=-1$. Back-substitution gives $x=3$. So solution is $(x,y)=(-3,1)$. Check by substituting $(x,y)=(-3,1)$ into the left & right sides. Results are $\begin{cases} Left=2(3)+3(-1)=3 \\ Left=(3)-(-1)=4 \end{cases}$ and $\begin{cases} Right=3 \\ Right=4 \end{cases}$

The left and right sides agree, so our solution is correct.

(b) $\begin{cases} 2x+6y=3 \\ 4x+12y=5 \end{cases}$ ➔ -2R₁+R₂ ➔ $\begin{cases} 2x+6y=3 \\ 0=-1 \end{cases}$ ➔ (1/2)R₁ ➔ $\begin{cases} x+3y=2 \\ 0=-1 \end{cases}$.. This system has no solutions.

(c) $\begin{cases} x-y-2z=2 \\ 4x-2y-2z=10 \end{cases}$ ➔ -4R₁+R₂ ➔ $\begin{cases} x-y-2z=2 \\ 2y+6z=2 \end{cases}$ ➔ (1/2)R₂ ➔ $\begin{cases} x-y-2z=2 \\ y+3z=1 \end{cases}$ This system has many solutions.

(d) $\begin{cases} 2x+2z=6 \\ x-y-2z=2 \\ 3x-y=8 \end{cases}$ ➔ R₁ ⇔ R₂ ➔ $\begin{cases} x-y-2z=2 \\ 2x+2z=6 \\ 3x-y=8 \end{cases}$ ➔ -2R₁+R₂ ➔ $\begin{cases} x-y-2z=2 \\ 2y+6z=2 \\ 3x-y=8 \end{cases}$ ➔ -3R₁+R₃ ➔

$\begin{cases} x-y-2z=2 \\ 2y+6z=2 \\ 2y+6z=2 \end{cases}$ ➔ -R₂+R₃ ➔ $\begin{cases} x-y-2z=2 \\ 2y+6z=2 \\ 0=0 \end{cases}$ ➔ (1/2)R₂ ➔ $\begin{cases} x-y-2z=2 \\ y+3z=1 \\ 0=0 \end{cases}$ System has many solutions.

[2] True or False: A system with three unknowns and two equations always has many solutions. You must justify your answer with a proof or a counterexample!

False! Counterexample: Observe that $\begin{cases} x+y+z=1 \\ 0=1 \end{cases}$ has no solutions.

[3] Solve each system using matrix notation. Express the solution set using vectors. Show all steps clearly. Check your solutions by substitution. (Show the check clearly.)

(a) $\begin{cases} 3x+2y=0 \\ 6x-2y+3z=18 \\ 3x-4y-z=18 \end{cases}$ ➔ matrix ➔ $\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 6 & -2 & 3 & 18 \\ 3 & -4 & -1 & 18 \end{array}\right)$ ➔ -2R₁+R₂ ➔ $\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -6 & 3 & 18 \\ 3 & -4 & -1 & 18 \end{array}\right)$ ➔ -R₁+R₃ ➔

$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -6 & 3 & 18 \\ 0 & -6 & -1 & 18 \end{array}\right)$ ➔ -R₂ + R₃ ➔ $\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -6 & 3 & 18 \\ 0 & 0 & -4 & 0 \end{array}\right)$ ➔ (-1/4)R₃ ➔ $\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -6 & 3 & 18 \\ 0 & 0 & 1 & 0 \end{array}\right)$ ➔ -3R₃ + R₂ ➔

$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -6 & 0 & 18 \\ 0 & 0 & 1 & 0 \end{array}\right)$ ➔ (-1/6)R₂ ➔ $\left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$ ➔ (-2)R₂+R₁ ➔ $\left(\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$ ➔ (1/3)R₁ ➔

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$ ➔ system ➔ $\begin{cases} x=2 \\ y=-3 \\ z=0 \end{cases}$. The solution set contains a single vector: $\left\{\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}\right\}$.

Check by substituting $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$ into the left and right sides of the original system. The results are

$\begin{cases} Left=3(2)+2(-3)=0 \\ Left=6(2)-2(-3)+3(0)=18 \\ Left=3(2)-4(-3)-(0)=18 \end{cases}$ and $\begin{cases} Right=0 \\ Right=18 \\ Right=18 \end{cases}$. Left and right sides agree, so our solution is correct.

(b) $\begin{cases} x+z=2 \\ 2x-y+3z=6 \\ 3x-y+4z=8 \end{cases}$ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & -1 & 3 & 6 \\ 3 & -1 & 4 & 8 \end{array}\right)$ ➔ -2R₁+R₂ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 3 & -1 & 4 & 8 \end{array}\right)$ ➔ -3R₁+R₃ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 2 \end{array}\right)$

➔ -R₂ + R₃ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ➔ (-1)R₂ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ➔ $\begin{cases} x+z=2 \\ y-z=-2 \\ 0=0 \end{cases}$. We see that z can be any real number. But to make the 1st equation true, we must have $x = 2 - z$ and to make the 2nd equation true, we must have $y = -2 + z$. The solution set can be written many ways.

$$\left\{\begin{pmatrix} 2-z \\ -2+z \\ z \end{pmatrix} \text{ such that } z \in \mathbb{R}\right\} = \left\{\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ z \end{pmatrix} \text{ such that } z \in \mathbb{R}\right\} = \left\{\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ such that } z \in \mathbb{R}\right\}$$

Check by substituting $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ -2+z \\ z \end{pmatrix}$ into the left and right sides of the original system.

The results are $\begin{cases} Left = (2-z)+z = 2 \\ Left = 2(2-z)-(-2+z)+3z = 4-2z+2-z+3z = 6 \\ Left = 3(2-z)-(-2+z)+4z = 6-3z+2-z+4z = 8 \end{cases}$ and $\begin{cases} Right = 2 \\ Right = 6 \\ Right = 8 \end{cases}$

The left and right sides agree, so our solution is correct.

(c) Observe that the system in (c) has the same left sides as the system in (b). The only difference is in the number on the right in the third equation. We suspect that there will be no solutions. Our approach should be to do exactly the same row operations that worked in (b).

$$\begin{cases} x+z=2 \\ 2x-y+3z=6 \\ 3x-y+4z=10 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & -1 & 3 & 6 \\ 3 & -1 & 4 & 10 \end{array}\right) \Rightarrow -2R_1+R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 3 & -1 & 4 & 10 \end{array}\right) \Rightarrow -3R_1+R_3 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 4 \end{array}\right)$$

$$\Rightarrow -R_2+R_3 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow (-1)R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow \begin{cases} x+z=2 \\ y-z=-2. \\ 0=2 \end{cases}$$

This system has no solutions, because no values of x, y, z will ever make the 3rd equation true.

[4] Decide if the vector is in the set. Justify your answers.

(a) Is vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in the set $\left\{\begin{pmatrix} -6 \\ 9 \end{pmatrix} k \middle| k \in \mathbb{R}\right\}$? Solution: Yes: let $k = -\frac{1}{3}$. Then $\begin{pmatrix} -6 \\ 9 \end{pmatrix} k = \begin{pmatrix} -6 \\ 9 \end{pmatrix}\left(-\frac{1}{3}\right) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) Is the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in the set $\left\{\begin{pmatrix} 4 \\ 6 \end{pmatrix} j \middle| j \in \mathbb{R}\right\}$? No. To make the 1st row match, we would need $j = 1/2$. But then the second row would not match because the equation $5 = 6j$ would not be true.

(c) Is the vector $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$ in the set $\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} j + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} k \middle| j, k \in \mathbb{R}\right\}$? No. To make the 2nd row match, we would need $k = -3$. To make the 3rd row match, we would need $j = 5$. But then the 1st row would not match because

$$3j + 2k = 3(5) + 2(-3) = 9 \neq 7$$

(d) Is the vector $\begin{pmatrix} 16 \\ 5 \\ 8 \end{pmatrix}$ in the set $\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} r \middle| r \in \mathbb{R}\right\}$?

Solution: Yes: let $r = 5$. Then $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} r = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}(5) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 15 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 8 \end{pmatrix}$

[5] Make up a three equations / three unknowns system having (a) no solutions. (b) exactly one solution. (c) a one-parameter solution set. (d) a two parameter solution set. Either explain your answers, or provide answers that are so clear that they do not need explanations.

Solution: (a) $\begin{cases} x=1 \\ y=2 \\ 0=3 \end{cases}$ has no solutions. (b) $\begin{cases} x=1 \\ y=2 \\ z=3 \end{cases}$ has one solution.

(c) $\begin{cases} x=1 \\ y=2 \\ 0=0 \end{cases}$ has a 1-parameter solution set with z as the parameter.

(d) $\begin{cases} x=1 \\ 0=0 \\ 0=0 \end{cases}$ has a 2-parameter solution set with y and z as the parameters.