[1] On Homework 1, you solved the four systems below. Now, express the solution set for each system using vectors. Identify the particular solution and the solution set of the homogeneous system. (You don't have to solve the systems again. Just rewrite the solution set.

(a) $\begin{cases} 2x+3y=3 \\ x-y=4 \end{cases}$ In Homework 1#[1](a), we found thesolution $(x,y)=(-3,1)$. So the solution set is $S=\left\{\begin{pmatrix}-3\\1\end{pmatrix}\right\}$. To identify particular and homogeneous parts, we write $S=\left\{\begin{pmatrix}-3\\1\end{pmatrix}\right\}=\left\{\underbrace{\begin{pmatrix}-3\\1\end{pmatrix}}_{particular}+\underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{homogeneous}\right\}$

(b) $\begin{cases} 2x+6y=3 \\ 4x+12y=5 \end{cases}$ In H1#[1](b), we found there were no solutions. So the solution set is the empty set, ϕ.

(c) $\begin{cases} x-y-2z=2 \\ 4x-2y-2z=10 \end{cases}$ In H1#[1](c), we did row operations to get the echelon form $\begin{cases} x-y-2z=2 \\ y+3z=1 \end{cases}$ and then simply concluded that the system has many solutions. In the current problem, we need to actually find the solution set. It will be helpful to do another step of row operations. $\begin{cases} x-y-2z=2 \\ y+3z=1 \end{cases}$ ➔ R_2+R_1 ➔ $\begin{cases} x+z=3 \\ y+3z=1 \end{cases}$

We see that z can be any real number, But the second equation requires that $y=1-3z$ and the first equation requires that $x=3-z$. So the solutions are of the form $\begin{pmatrix}3-z\\1-3z\\z\end{pmatrix}$ where z is any real number.

Check by substituting $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3-z\\1-3z\\z\end{pmatrix}$ into the left and right sides of the original system. The results are

$$\begin{cases} Left=(3-z)-(1-3z)-2(z)=3-z-1+3z-2z=2 \\ Left=4(3-z)-2(1-3z)-2(z)=12-4z-2+6z-2z=10 \end{cases} \text{ and } \begin{cases} Right=2 \\ Right=10 \end{cases}.$$

Left and right sides agree, so our solution is correct. The solution set is $S=\left\{\underbrace{\begin{pmatrix}3\\1\\0\end{pmatrix}}_{particular}+\underbrace{\begin{pmatrix}-1\\-3\\1\end{pmatrix}z \,\middle|\, z\in\mathbb{R}}_{homogeneous}\right\}$.

(d) $\begin{cases} 2x+2z=6 \\ x-y-2z=2 \\ 3x-y=8 \end{cases}$ This part (d) is very similar to part (c). In H1#[1](d), we did row operations to get the echelon form $\begin{cases} x-y-2z=2 \\ y+3z=1 \\ 0=0 \end{cases}$ and then simply concluded that the system has many solutions. In the current problem, we need to actually give the solution set. For that, it will be helpful to do the same additional step of row operations that we used in part (c), above:

$$\begin{cases} x-y-2z=2 \\ y+3z=1 \\ 0=0 \end{cases} \text{ ➔ } R_2+R_1 \text{ ➔ } \begin{cases} x+z=3 \\ y+3z=1. \\ 0=0 \end{cases}$$

The solution set will be the same as the solution set to part (c). That is, $S=\left\{\underbrace{\begin{pmatrix}3\\1\\0\end{pmatrix}}_{particular}+\underbrace{\begin{pmatrix}-1\\-3\\1\end{pmatrix}z \,\middle|\, z\in\mathbb{R}}_{homogeneous}\right\}$.

[2] (20 points) (Similar to One.I.3#16) For the system $\begin{cases} x+2y-z+3w=1 \\ 2x+4y-z+6w=-2 \\ y+2w=-3 \end{cases}$, which of the vectors can be used as the particular solution part of some general solution? Show all steps clearly and explain clearly.

(a) We **check** by substituting $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -4 \\ 0 \end{pmatrix}$ into the left and right sides of the original system. The results are

$$\begin{cases} Left = (3) + 2(-3) - (-4) + 3(0) = 1 \\ Left = 2(3) + 4(-3) - (-4) + 6(0) = -2 \\ Left = (-3) + 2(0) = -3 \end{cases} \text{ and } \begin{cases} Right = 1 \\ Right = -2 \\ Right = -3 \end{cases}.$$

Since $Left = Right$, we conclude that the vector is a particular solution.

(b) We **check** by substituting $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$ into the left and right sides of the original system. The results are

$$\begin{cases} Left = (-1) + 2(-1) - (1) + 3(3) = 5 \\ Left = 2(-1) + 4(-1) - (1) + 6(3) = 11 \\ Left = (-1) + 2(3) = 5 \end{cases} \text{ and } \begin{cases} Right = 1 \\ Right = -2 \\ Right = -3 \end{cases}.$$

Since $Left \neq Right$, we conclude that the vector is not a particular solution.

(c) We **check** by substituting $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \\ -1 \end{pmatrix}$ into the left and right sides of the original system. The results are

$$\begin{cases} Left = (2) + 2(-1) - (-4) + 3(-1) = 1 \\ Left = 2(2) + 4(-1) - (-4) + 6(-1) = -2 \\ Left = (-1) + 2(-1) = -3 \end{cases} \text{ and } \begin{cases} Right = 1 \\ Right = -2 \\ Right = -3 \end{cases}.$$

Since $Left = Right$, we conclude that the vector is not a particular solution.

[3] Lemma 3.7 says that we can use any particular solution for $\vec{p}$.

Find, if possible, a general solution to the system $\begin{cases} x - y + 2z = -5 \\ 3x + y + 2z = -3 \\ 2x + y + z = -1 \end{cases}$

that uses the given vector at right as its particular solution.
Show all steps clearly and explain clearly.

(a) $\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Solution:

Part 1: Which vectors qualify? We investigate which of the vectors (a),(b),(c) qualify as particular solutions. We do this by substituting each one into the original system.

We check (a) by substituting the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ into the system. The result is

$$\begin{cases} Left = (-3) - (4) + 2(1) = -5 \\ Left = 3(-3) + (4) + 2(1) = -3 \\ Left = 2(-3) + (4) + (1) = -1 \end{cases} \text{ and } \begin{cases} Right = -5 \\ Right = -3 \\ Right = -1 \end{cases}$$

Since $Left = Right$, the vector is a particular solution. Therefore, it can be used to build a general solution.

We check (b) by substituting the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ into the system. The result is

$$\begin{cases} Left = (-1) - (2) + 2(-1) = -5 \\ Left = 3(-1) + (2) + 2(-1) = -3 \\ Left = 2(-1) + (2) + (-1) = -1 \end{cases} \text{ and } \begin{cases} Right = -5 \\ Right = -3 \\ Right = -1 \end{cases}$$

Since $Left = Right$, the vector is a particular solution. Therefore, it can be used to build a general solution.

We check (c) by substituting $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ into the system. The result is

$$\begin{cases} Left = (1) - (-2) + 2(3) = 9 \\ Left = 3(1) + (-2) + 2(3) = 7 \\ Left = 2(1) + (-2) + (3) = 3 \end{cases} \text{ and } \begin{cases} Right = -5 \\ Right = -3 \\ Right = -1 \end{cases}$$

Since $Left \neq Right$, the vector is not a particular solution. It cannot be used to build a general solution.

Part 2: Find the solution of the associated homogeneous system. In order to build general solutions using the vectors from (a) and (b), we will need to know the solution of the associated homogeneous system.

$$\begin{cases} x - y + 2z = 0 \\ 3x + y + 2z = 0 \\ 2x + y + z = 0 \end{cases} ➔ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) ➔ \text{-3}R_1\text{+}R_2 ➔ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 4 & -4 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) ➔ \text{-3}R_1\text{+}R_3 ➔ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) ➔$$

$$(1/4)R_2 ➔ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) ➔ R_2\text{+}R_1 ➔ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) ➔ \text{(-3)}R_2\text{+}R_3 ➔ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ➔ \begin{cases} x + z = 0 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

We see that z can be any real number, as long as $x = -z$ and $y = z$. So all vectors of the form $\begin{pmatrix} -z \\ z \\ z \end{pmatrix}$.

Check by substituting the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix}$ into the homogeneous system. The result is

$$\begin{cases} Left = (-z) - (z) + 2(z) = 0 \\ Left = 3(-z) + (z) + 2(z) = 0 \\ Left = 2(-z) + (z) + (z) = 0 \end{cases} \text{ and } \begin{cases} Right = 0 \\ Right = 0 \\ Right = 0 \end{cases}$$

Since $Left = Right$, we know that the vector is a solution to the homogeneous system.

In parametrized form, the solution set for the homogeneous system is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z \,\middle|\, z \in \mathbb{R} \right\}$.

Part 3: Build the general solutions using the particular solution vectors from (a) and (b).

The general solution using the vector from (a) is $S = \left\{ \underbrace{\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}}_{particular} + \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z \,\middle|\, z \in \mathbb{R}}_{homogeneous} \right\}$.

The general solution using the vector from (b) is $S = \left\{ \underbrace{\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}_{particular} + \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z \,\middle|\, z \in \mathbb{R}}_{homogeneous} \right\}$.

[4] (20 points) (Similar to One.I.3#19) Singular or nonsingular? Show all steps clearly and explain clearly.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ➔ Using row reduction to reduced echelon form ➔ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix does not have a row of all zeroes, so the original matrix is non-singular.

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 10 \end{pmatrix}$ The matrix is not square, so the terminology of singular/nonsingular does not apply.

(c) $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ The matrix is in echelon form, and it has a row of all zeroes. Conclude that it is singular.

(d) $\begin{pmatrix} 4 & 5 & 1 \\ 1 & 0 & 5 \\ -1 & 1 & 2 \end{pmatrix}$ ➔ Using row reduction to reduced echelon form ➔ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This matrix does not have a row of all zeroes, so the original matrix is non-singular.

(e) $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & 7 & 7 \end{pmatrix}$ ➔ row reduction ➔ $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Has row of all zeroes, so original matrix is singular.

[5] Is the vector $\begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$ in the set generated by the set $\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right\}$?

Solution: Define vector $\vec{v} = \begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$ and define set $S = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right\}$. We are asked if the vector $\vec{v}$ is in the set generated by S. In other words, we are being asked if there exists a linear combination of the three vectors $\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}$ that equals vector $\vec{v}$. That is, we are being asked if there exist real number constants c_1, c_2, c_3 such that $c_1\overrightarrow{w_1} + c_2\overrightarrow{w_2} + c_3\overrightarrow{w_3} = \vec{v}$. That is, $c_1 \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$.

Doing the scalar multiplication, this equation becomes $\begin{pmatrix} 2c_1 \\ 0c_1 \\ 3c_1 \end{pmatrix} + \begin{pmatrix} 1c_2 \\ -5c_2 \\ 7c_2 \end{pmatrix} + \begin{pmatrix} 4c_3 \\ 1c_3 \\ -2c_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$

Doing the vector addition, this equation becomes $\begin{pmatrix} 2c_1 + 1c_2 + 4c_3 \\ 0c_1 - 5c_2 + 1c_3 \\ 3c_1 + 7c_2 - 2c_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$

This vector equation amounts to a system of linear equations that we can solve for the three constants c_1, c_2, c_3.

$\begin{cases} 2c_1 + 1c_2 + 4c_3 = 12 \\ 0c_1 - 5c_2 + 1c_3 = -7 \\ 3c_1 + 7c_2 - 2c_3 = 5 \end{cases}$ ➔ matrix ➔ $\left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -5 & 1 & -7 \\ 3 & 7 & -2 & 5 \end{array}\right)$ ➔ -1R₁+R₃ ➔ $\left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -5 & 1 & -7 \\ 1 & 6 & -6 & -7 \end{array}\right)$ ➔ -2R₃+R₁

➔ $\left(\begin{array}{ccc|c} 0 & -11 & 16 & 26 \\ 0 & -5 & 1 & -7 \\ 1 & 6 & -6 & -7 \end{array}\right)$ ➔ R₁⇔R₃ ➔ $\left(\begin{array}{ccc|c} 1 & 6 & -6 & -7 \\ 0 & -5 & 1 & -7 \\ 0 & -11 & 16 & 26 \end{array}\right)$ ➔ 2R₂ ➔ $\left(\begin{array}{ccc|c} 1 & 6 & -6 & -7 \\ 0 & -10 & 2 & -14 \\ 0 & -11 & 16 & 26 \end{array}\right)$ ➔ -R₃+R₂

➔ $\left(\begin{array}{ccc|c} 1 & 6 & -6 & -7 \\ 0 & 1 & -14 & -40 \\ 0 & -11 & 16 & 26 \end{array}\right)$ ➔ 11R₂+R₃ ➔ $\left(\begin{array}{ccc|c} 1 & 6 & -6 & -7 \\ 0 & 1 & -14 & -40 \\ 0 & 0 & -138 & -414 \end{array}\right)$ ➔ (-1/138)R₃ ➔ $\left(\begin{array}{ccc|c} 1 & 6 & -6 & -7 \\ 0 & 1 & -14 & -40 \\ 0 & 0 & 1 & 3 \end{array}\right)$

➔ 14R₃+R₂ ➔ 6R₃+R₁ ➔ $\left(\begin{array}{ccc|c} 1 & 6 & 0 & 11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ➔ -6R₂+R₁ ➔ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ➔ system ➔ $\begin{cases} c_1 = -1 \\ c_2 = 2 \\ c_3 = 3 \end{cases}$

Check by substituting $(c_1, c_2, c_3) = (-1,2,3)$ into the left and right sides of the original system. The results are

$\begin{cases} Left = 2(-1) + 1(2) + 4(3) = 12 \\ Left = 0(-1) - 5(2) + 1(3) = -7 \\ Left = 3(-1) + 7(2) - 2(3) = 5 \end{cases}$ and $\begin{cases} Right = 12 \\ Right = -7 \\ Right = 5 \end{cases}$. Left and right agree: our solution is correct.

We need to write a clear conclusion, though. (The original question did not ask us to find c_1, c_2, c_3.)

Conclusion: The vector $\begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$ is in the set generated by the set $\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right\}$ because

$$(-1) \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + (2) \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} + (3) \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ -7 \\ 5 \end{pmatrix}$$