[1] Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 | x - 5y + 7z = 0 \right\}$. Show that S is a vector space. (Use only Section Two.I.1 concepts. That is, prove that each of the ten vector space conditions is satisfied.)

Prove that Condition (1) is satisfied.

Suppose that $\vec{v}$ and $\vec{w}$ are elements of S. So $\vec{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ such that $x_1 - 5y_1 = 7z_1 = 0$ and $\vec{w} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ such that $x_2 - 5y_2 + 7z_2 = 0$. We must show that $\vec{v} + \vec{w} \in S$.

The sum $\vec{v} + \vec{w} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ is an element of $\mathbb{R}^3$. (We already know that $\mathbb{R}^3$ is a vector space.) We must show that the vector $\vec{v} + \vec{w}$ is also an element of S. That is, we must show that $\vec{v} + \vec{w}$ satisfies the extra requirement. Observe that

$$(x_1 + x_2) - 5(y_1 + y_2) + 7(z_1 + z_2) = (x_1 - 5y_1 + 7z_1) + (x_2 - 5y_2 + 7z_2) = 0 + 0 = 0$$

So $\vec{v} + \vec{w}$ is an element of $\mathbb{R}^3$ and it also satisfies the extra requirement. That is, $\vec{v} + \vec{w}$ is an element of S. Conclude that Condition (1) is satisfied.

Prove that Condition (2) is satisfied.

Suppose that $\vec{v}$ and $\vec{w}$ are elements of S. Addition is commutative for all vectors in $\mathbb{R}^3$, so it is commutative for vectors that also happen to be in S. That is, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$. Conclude that Condition (2) is satisfied.

Prove that Condition (3) is satisfied.

Suppose that $\vec{u}, \vec{v}, \vec{w}$ are elements of S. Addition is associative for all vectors in $\mathbb{R}^3$, so it is associative for vectors that also happen to be in S. That is, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$. So Condition (3) is satisfied.

Prove that Condition (4) is satisfied.

We know that there is a zero element $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in the vector space $\mathbb{R}^3$. We must show that $\vec{0}$ is also in S.

To do that, we must show that $\vec{0}$ satisfies the additional requirement. Observe that

$$(0) - 5(0) + 7(0) = 0 + 0 + 0 = 0$$

So the zero element $\vec{0}$ in the vector space $\mathbb{R}^3$ satisfies the additional requirement to be an element of S. Conclude that Condition (4) is satisfied.

Prove that Condition (5) is satisfied.

Suppose that $\vec{v}$ is an element of S. So $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x - 5y + 7z = 0$. The vector $\vec{w} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$ is the additive inverse of $\vec{v}$ in the vector space $\mathbb{R}^3$. It is We must show that $\vec{w}$ is also in S. To do that, we must show that $\vec{w}$ satisfies the additional requirement. Observe that

$$(-x) - 5(-y) + 7(z) = -(x - 5y + 7z) = -0 = 0$$

So the vector $\vec{w}$ in the vector space $\mathbb{R}^3$ satisfies the additional requirement to be an element of S. Conclude that Condition (5) is satisfied.

Prove that Condition (6) is satisfied.

Suppose that $\vec{v}$ is an element of S and that $r \in \mathbb{R}$ is some scalar. So $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x - 5y + 7z = 0$.

We already know that $r \cdot \vec{v} \in \mathbb{R}^3$, because $\mathbb{R}^3$ is a vector space. We must show that $r \cdot \vec{v}$ is also in S. To do that, we must show that $r \cdot \vec{v}$ satisfies the additional requirement. Note that $r \cdot \vec{v} = r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$. Now note that $(rx) - 5(ry) + 7(rz) = r(x - 5y + 7z) = r0 = 0$. So the vector $r \cdot \vec{v}$ in the vector space $\mathbb{R}^3$ satisfies the additional requirement to be an element of S. Conclude that Condition (6) is satisfied.

Prove that Condition (7) is satisfied.

Suppose that $\vec{v}$ is an element of S and that $r, s \in \mathbb{R}$ are scalars. The equation $(r+s)\cdot\vec{v} = r\cdot\vec{v} + s\cdot\vec{v}$ is true for all vectors $\vec{v}$ in $\mathbb{R}^3$, so it is true for vectors $\vec{v}$ that are also in S. So Condition (7) is satisfied.

Prove that Condition (8) is satisfied.

Suppose that $\vec{v}, \vec{w}$ are elements of S and that $r \in \mathbb{R}$ is a scalar. The equation $r\cdot(\vec{v}+\vec{w}) = r\cdot\vec{v} + r\cdot\vec{w}$ is true for all vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^3$, so it is true for vectors $\vec{v}, \vec{w}$ in S. Conclude that Condition (8) is satisfied.

Prove that Condition (9) is satisfied.

Suppose that $\vec{v}$ is an element of S and that $r, s \in \mathbb{R}$ are scalars. The equation $(rs)\cdot\vec{v} = r\cdot(s\cdot\vec{v})$ is true for all vectors $\vec{v}$ in $\mathbb{R}^3$, so it is true for vectors $\vec{v}$ in S. Conclude that Condition (9) is satisfied.

Prove that Condition (10) is satisfied.

Suppose that $\vec{v}$ is an element of S. The equation $1\cdot\vec{v} = \vec{v}$ is true for all vectors $\vec{v}$ in $\mathbb{R}^3$, so it is true for vectors $\vec{v}$ in S. Conclude that Condition (10) is satisfied.

Conclusion: Since all ten conditions are satisfied, we conclude that S is a vector space.

[2] Is the set $\mathbb{Z}$ of integers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

- If you say that it is, then prove that each of the ten vector space conditions is satisfied.
- If you say that it is not, then explain which vector space conditions are not satisfied.

This is not a vector space because it fails condition (6). To see why, consider the vector $\vec{v} = 3 \in \mathbb{Z}$ and the scalar $r = \frac{1}{2} \in \mathbb{R}$. Observe that $r\cdot\vec{v} = \frac{1}{2}(3) = \frac{3}{2} \notin \mathbb{R}$. So the set $\mathbb{Z}$ is not closed under the operation of scalar multiplication by scalars in $\mathbb{R}$.

[3] In Example 1.12, on page 84, we were introduced to the following set, along with an operation of addition and an operation of scalar multiplication:

- Let $\mathcal{F} = \{f | f: \mathbb{R} \to \mathbb{R}\}$. That is, $\mathcal{F}$ is the set of real-valued functions of one real variable.
- For two functions f, g the symbol $f + g$ denotes the function defined by $(f+g)(x) = f(x) + g(x)$.
- For a function f and a real number r the symbol $r\cdot f$ denotes the function defined by $(r\cdot f)(x) = rf(x)$.

The author states that with those operations, the set $\mathcal{F}$ of real valued functions is a vector space, but he does not verify that any of the vector space conditions are satisfied.

Prove that the set $\mathcal{F}$, with the operations described, does qualify to be called a vector space. That is, prove that all ten of the vector space conditions are satisfied.

Prove that Condition (1) is satisfied.

Suppose that f and g are elements of $\mathcal{F}$. So f is a function $f: \mathbb{R} \to \mathbb{R}$ and g is a function $g: \mathbb{R} \to \mathbb{R}$.
We must show that $f + g \in \mathcal{F}$.

By the definition of function addition given above, the symbol $f + g$ denotes the function defined by $(f+g)(x) = f(x) + g(x)$. We have to figure out the domain and codomain.

Notice that this function called $f + g$ takes as input the same sort of input x that can be used as in put to the individual functions f and g. That is, the domain of $f + g$ is the set $\mathbb{R}$ of all real numbers.

To figure out the codomain of the function $f + g$, notice that the output $(f+g)(x)$ is the sum $f(x) + g(x)$. But the symbols $f(x)$ and $g(x)$ both represent real numbers (because function $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$). So their sum $f(x) + g(x)$ will be a real number. Conclude that the output of the function $f + g$ will be a real number. So the codomain of the function $f + g$ will be the set $\mathbb{R}$ of real numbers.

Since the domain and codomain of the function $f + g$ are $\mathbb{R}$, we write $(f+g): \mathbb{R} \to \mathbb{R}$. So $f + g \in \mathcal{F}$.

Conclude that Condition (1) is satisfied.

Prove that Condition (2) is satisfied.

Suppose that f and g are elements of $\mathcal{F}$. We must show that $f+g=g+f$.

The equation $f+g=g+f$ is a statement about equality of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$(f+g)(x) = f(x)+g(x)$ by definition of function addition
$= g(x)+f(x)$ because real number addition is commutative
$= (g+f)(x)$ by definition of function addition.

This proves that $f+g=g+f$. Conclude that Condition (2) is satisfied.

Prove that Condition (3) is satisfied.

Suppose f,g,h are elements of $\mathcal{F}$. We must show that $(f+g)+h=f+(g+h)$.
The equation $(f+g)+h=f+(g+h)$ is an equation of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$\big((f+g)+h\big)(x) = (f+g)(x)+h(x)$ by definition of function addition
$= \big(f(x)+g(x)\big)+h(x)$ by definition of function addition
$= f(x)+\big(g(x)+h(x)\big)$ because real number addition is associative
$= f(x)+(g+h)(x)$ by definition of function addition.
$= \big(f+(g+h)\big)(x)$ by definition of function addition.

This proves that $(f+g)+h=f+(g+h)$. Conclude that Condition (3) is satisfied.

Prove that Condition (4) is satisfied.

We must show that the set $\mathcal{F}$ contains a zero vector. That is, we must show that there is a function $\vec{0}\in\mathcal{F}$ such that for all functions $f\in\mathcal{F}$, the equation $f+\vec{0}=0$ is true. We must come up with such a function $\vec{0}$. Define the function $\vec{0}:\mathbb{R}\to\mathbb{R}$ by saying that for any real number input x,the output $\vec{0}(x)$ will be the real number 0. That is, $\vec{0}(x)=0$ for all $x\in\mathbb{R}$.

We must start by confirming that $\vec{0}\in\mathcal{F}$. Notice that the domain of $\vec{0}$ is $\mathbb{R}$ and that the domain of $\vec{0}$ is $\mathbb{R}$ as well. So $\vec{0}\in\mathcal{F}$.

It is easy to confirm that the zero function $\vec{0}$ behaves as a zero element should in a vector space. Let f be any function $f:\mathbb{R}\to\mathbb{R}$. We must show that $f+\vec{0}=f$. This is an equation of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$\big(f+\vec{0}\big)(x) = f(x)+\vec{0}(x)$ by definition of function addition
$= f(x)+0$ by definition of how the zero function works
$= f(x)$ by definition of function addition.

This proves that $f+\vec{0}=f$.

Conclude that the set $\mathcal{F}$ contains a zero vector, so Condition (4) is satisfied.

Prove that Condition (5) is satisfied.

We must show that $\mathcal{F}$ contains an additive inverse for each of its elements.
Suppose that f is an element of $\mathcal{F}$. So f is a function $f:\mathbb{R}\to\mathbb{R}$.
We must show that there exists a function $g\in\mathcal{F}$ such that $f+g=\vec{0}$.
Let the function g be defined by $g(x)=-f(x)$.
We must start by confirming that $g\in\mathcal{F}$.
Observe that g takes the same kind of input that f does, so the domain of g is the set $\mathbb{R}$ of real numbers. Also observe that the output of g is the negative of the output of f. Since the function f outputs real numbers, this tells that the function g also outputs real numbers. So we can write $g:\mathbb{R}\to\mathbb{R}$. Hence, $g\in\mathcal{F}$.

Now we must show that g has the correct behavior. That is, we must show that $f+g=\vec{0}$. This is a statement about equality of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$(f+g)(x) = f(x)+g(x)$ by definition of function addition

$= f(x) + \big(-f(x)\big)$ by definition of how the function g works
$= 0$ by definition of the additive inverse of a real number.
$= \vec{0}(x)$ by definition of how the zero function $\vec{0}$ works

This proves that $f + g = \vec{0}$. So the function g is an additive inverse for f.

Conclude that the set $\mathcal{F}$ contains an additive inverse for each element, so Condition (5) is satisfied.

Prove that Condition (6) is satisfied.

Suppose that f is an element of $\mathcal{F}$ and that $r \in \mathbb{R}$ is some scalar. We must show that $r \cdot f \in \mathcal{F}$. By the definition of scalar multiplication of functions addition given above, the symbol $r \cdot f$ denotes the function defined by $(r \cdot f)(x) = rf(x)$. We have to figure out the domain and codomain.

Notice that this function called $r \cdot f$ takes as input the same sort of input x that can be used as input to the individual function f. That is, the domain of $r \cdot f$ is the set $\mathbb{R}$ of all real numbers.
Now, to figure out the codomain of the function $r \cdot f$, notice that the output $(r \cdot f)(x)$ is $rf(x)$. But the symbols r and $f(x)$ both represent real numbers (because r is a scalar and function $f\colon \mathbb{R} \to \mathbb{R}$). So their product $rf(x)$ will be a real number. Conclude that the output of the function $r \cdot f$ will be a real number. So the codomain of the function $r \cdot f$ will be the set $\mathbb{R}$ of real numbers.

Since the domain and codomain of the function $r \cdot f$ are the set $\mathbb{R}$, we write $(r \cdot f)\colon \mathbb{R} \to \mathbb{R}$. So $r \cdot f \in \mathcal{F}$.
Conclude that Condition (6) is satisfied.

Prove that Condition (7) is satisfied.

Suppose that f is an element of $\mathcal{F}$ and that $r, s \in \mathbb{R}$ are scalars.
We must prove that the equation $(r+s) \cdot f = r \cdot f + s \cdot f$ is true. The equation $(r+s) \cdot f = r \cdot f + s \cdot f$ is an equality of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$\big((r+s) \cdot f\big)(x) = (r+s)f(x)$ by definition of scalar multiplication of functions
$= rf(x) + sf(x)$ by distributive property for real numbers
$= (r \cdot f)(x) + (s \cdot f)(x)$ by definition of scalar multiplication of functions
$= (r \cdot f + s \cdot f)(x)$ by definition of addition of functions

This proves that $(r+s) \cdot f = r \cdot f + s \cdot f$. **Conclude** that Condition (7) is satisfied.

Prove that Condition (8) is satisfied.

Suppose that f, g are elements of $\mathcal{F}$ and that $r \in \mathbb{R}$ is a scalar. We must prove that the equation $r \cdot (f+g) = r \cdot f + r \cdot g$ is true. The equation $r \cdot (f+g) = r \cdot f + r \cdot g$ is an equation of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$\big(r \cdot (f+g)\big)(x) = r(f+g)(x)$ by definition of scalar multiplication of functions
$= r\big(f(x) + g(x)\big)$ by definition of function addition
$= rf(x) + rg(x)$ by distributive property for real numbers
$= (r \cdot f)(x) + (r \cdot g)(x)$ by definition of scalar multiplication of functions
$= (r \cdot f + r \cdot g)(x)$ by definition of addition of functions

This proves that $r \cdot (f+g) = r \cdot f + r \cdot g$. **Conclude** that Condition (8) is satisfied.

Prove that Condition (9) is satisfied.

Suppose that f is an element of S and that $r, s \in \mathbb{R}$ are scalars.
We must prove that the equation $(rs) \cdot f = r \cdot (s \cdot f)$ is true. The equation $(rs) \cdot f = r \cdot (s \cdot f)$ is an equation of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$\big((rs) \cdot f\big)(x) = (rs)f(x)$ by definition of scalar multiplication of functions
$= r\big(sf(x)\big)$ by the associative property of real number multiplication
$= r\big((s \cdot f)(x)\big)$ by definition of scalar multiplication of functions
$= \big(r \cdot (s \cdot f)\big)(x)$ by definition of scalar multiplication of functions

This proves that $(rs) \cdot f = r \cdot (s \cdot f)$. **Conclude** that Condition (9) is satisfied.

Prove that Condition (10) is satisfied.

Suppose that f is an element of $\mathcal{F}$.

We must prove that the equation $1 \cdot f = f$ is true.

The equation $1 \cdot f = f$ is a statement about equality of functions. We show that functions are equal by showing that they always produce equal outputs when fed equal inputs.

$$\begin{aligned} (1 \cdot f)(x) &= 1f(x) \text{ by definition of scalar multiplication of functions} \\ &= f(x) \text{ by property of the real number 1} \end{aligned}$$

This proves that $1 \cdot f = f(rs) \cdot f = r \cdot (s \cdot f)$. Conclude that Condition (10) is satisfied.

Conclusion: We have shown that all ten vector space conditions are satisfied. Conclude $\mathcal{F}$ is a vector space.

[4] Let V be the vector space of 2×2 matrices with real entries, with operations of matrix addition and scalar multiplication by real numbers. Let $UT = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b \in \mathbb{R} \right\}$. This set UT is called the *upper triangular* 2×2 matrices. The set UT is a subset of V, so it inherits the operations of addtion and scalar multiplication by real numbers. Is set UT a subspace? Explain clearly.

Solution: Notice that UT is a subset of the vector space $\mathcal{M}_{2\times 2}$ of 2×2 matrices. So we don't have to check all ten conditions to prove that S is a vector space. (This is the subject of Section Two.I.2)

Observe that UT can be written $S = \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$. This tells us that S is the span of the set $T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathcal{M}_{2\times 2}$. That is, $S = [T] = \left[\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right]$.

Since T is a subset of the vector space, $T \subset \mathcal{M}_{2\times 2}$, Lemma 2.15, on page 95 tells us that $[T]$ will be a SUBSPACE of $\mathcal{M}_{2\times 2}$. That is, $S = [T]$ will be a SUBSPACE of $\mathcal{M}_{2\times 2}$.

[5] The *odd* functions are defined as the set $Odd = \{f: \mathbb{R} \to \mathbb{R} | f(-x) = -f(x)\}$. This is a subset of the set of all real-valued functions, and so it inherits the operations of function addition and scalar multiplication by real numbers. (Those operations were defined above in problem [3].) Is the subset Odd a subspace? Explain clearly. (Your explanation will need more detail than the book's answer to a similar question.)

Solution: I will show that Odd is a vector space.

Observe that $Odd \subset \mathcal{F}$, the vector space of all real-valued functions. So we don't have to check all ten conditions to prove that Odd is a vector space. (This is the subject of Section Two.I.2) I will provide two different solution methods:

Solution A: Check only the conditions that need to be checked.

Prove that Condition (1) is satisfied.

Suppose that f and g are elements of S.

So f is a function $f: \mathbb{R} \to \mathbb{R}$ such that satisfies the extra requirement $f(-x) = -f(x)$ and g is a function $g: \mathbb{R} \to \mathbb{R}$ such that satisfies the extra requirement $g(-x) = -g(x)$

We must show that $f + g \in S$.

The sum $f + g$ is a function $(f + g): \mathbb{R} \to \mathbb{R}$ (by the definition of function addition given above.) We must show that $f + g$ is also an element of S. That is, we must show that $f + g$ satisfies the extra requirement.

Observe that

$$\begin{aligned} (f + g)(-x) &= f(-x) + g(-x) \text{ by definition of function addition} \\ &= -f(x) - g(x) \text{ because } f(-x) = -f(x) \text{ and } g(-x) = -g(x) \\ &= -\big(f(x) + g(x)\big) \text{ by arithmetic} \\ &= -(f + g)(x) \text{ by definition of function addition} \end{aligned}$$

So $f+g$ is a function $(f+g)\colon \mathbb{R} \to \mathbb{R}$ and it also satisfies the extra requirement. That is, $f+g$ is an element of S. Conclude that Condition (1) is satisfied.

Conditions (2), (3) are automatically satisfied because $\boldsymbol{Odd} \subset \boldsymbol{\mathcal{F}}$.

Prove that Condition (4) is satisfied.

We already know from class that the vector space $\mathcal{F}$ of all functions has a zero element. It is the zero function $\vec{0}\colon \mathbb{R} \to \mathbb{R}$ defined by the formula $\vec{0}(x) = 0$. (This was discussed in problem [3] above.)

For the current problem, we must show that the zero function $\vec{0}$ is also an element of S. To do that, we simply observe that the zero function $\vec{0}$ also has the extra property that $\vec{0}(-x) = 0 = -0 = -\vec{0}(x)$. So $\vec{0}$ is an element of S. Conclude that Condition (4) is satisfied.

Prove that Condition (5) is satisfied.

Suppose that f is an element of S. So f is a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(-x) = -f(x)$ We already know that f has an additive inverse in the vector space $\mathcal{F}$ of all functions $\mathcal{F} = \{f\colon \mathbb{R} \to \mathbb{R}\}$. It is the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -f(x)$. We must show that g is also in S. To do that, we must show that g satisfies the additional requirement. Observe that

$$
\begin{aligned}
g(-x) &= -f(-x) \text{ by definition of how } g \text{ works} \\
&= -\big(-f(x)\big) \text{ because } f(-x) = -f(x) \\
&= -g(x) \text{ by the definition of how } g \text{ works.}
\end{aligned}
$$

So the function g in the larger vector space $\mathcal{F}$ satisfies the additional requirement to be an element of S. Thus, every function in S has an additive inverse in S. Conclude that Condition (5) is satisfied.

Prove that Condition (6) is satisfied.

Suppose that f is an element of S and that $r \in \mathbb{R}$ is some scalar. We must show that $r \cdot f$ is also an element of S. We know that $r \cdot f$ is a function, $(r \cdot f)\colon \mathbb{R} \to \mathbb{R}$, by the definition of scalar multiplication of functions. That is, we know that $r \cdot f$ is an element of the vector space $\mathcal{F}$ of all functions. We must also show that $r \cdot f$ satisfies the additional requirement to be an element of S. We check:

$$
\begin{aligned}
(r \cdot f)(-x) &= r\big(f(-x)\big) \text{ by definition of scalar multiplication of functions} \\
&= r\big(-f(x)\big) \text{ because } f(-x) = -f(x) \\
&= -r\big(f(x)\big) \text{ by arithmetic.} \\
&= -(r \cdot f)(x) \text{ by definition of scalar multiplication of functions.}
\end{aligned}
$$

So the function $r \cdot f$ satisfies the additional requirement to be an element of S. Conclude that Condition (6) is satisfied.

Conditions (7), (8), (9), (10) are automatically satisfied because $\boldsymbol{Odd} \subset \boldsymbol{\mathcal{F}}$.

Conclude that the set Odd is a vector space because it is a subspace of $\mathcal{F}$.

End of Solution A

Solution B: Use Lemma 2.9 on page 92

Let $f, g \in S$ and $r_1, r_2 \in \mathbb{R}$. We will show that $r_1 \cdot f + r_2 \cdot g \in S$. To do that, we must show that $r_1 \cdot f + r_2 \cdot g$ is an *odd* function. That is, we must show that $(r_1 \cdot f + r_2 \cdot g)(-x) = -(r_1 \cdot f + r_2 \cdot g)(x)$.

$$
\begin{aligned}
(r_1 \cdot f + r_2 \cdot g)(-x) &= (r_1 \cdot f)(-x) + (r_2 \cdot g)(-x) \text{ by definition of function addition} \\
&= r_1\big(f(-x)\big) + r_2\big(g(-x)\big) \text{ by definition of scalar multiplication} \\
&= r_1\big(-f(x)\big) + r_2\big(-g(x)\big) \text{ because } f \text{ and } g \text{ are odd functions} \\
&= -r_1\big(f(x)\big) - r_2\big(g(x)\big) \text{ by arithmetic.} \\
&= -\Big(r_1\big(f(x)\big) + r_2\big(g(x)\big)\Big) \text{ by arithmetic.} \\
&= -\big((r_1 \cdot f)(x) + (r_2 \cdot g)(x)\big) \text{ by definition of scalar multiplication} \\
&= -\big((r_1 \cdot f + r_2 \cdot g)(x)\big) \text{ by definition of function addition} \\
&= -(r_1 \cdot f + r_2 \cdot g)(x) \text{ eliminated one pair of parentheses}
\end{aligned}
$$

Now Lemma 2.9 on page 92 tells us that set Odd is a subspace of $\mathcal{F}$.

End of Solution B