[1] (A) We must determine if there exists a non-trivial linear combination of $\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}$ that equals $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

That is, we want to determine if there exist $x, y, z \in \mathbb{R}$, not all zero, such that $x\begin{pmatrix}2\\2\\1\end{pmatrix} + y\begin{pmatrix}5\\4\\3\end{pmatrix} + z\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

That is, we want to know if there is a solution to the system $\begin{cases}2x + 5y + 1z = 0\\ 2x + 4y - 1z = 0\\ 1x + 3y + 1z = 0\end{cases}$ This system is represented by augmented matrix $\left(\begin{array}{ccc|c}2 & 5 & 1 & 0\\ 2 & 4 & -1 & 0\\ 1 & 3 & 1 & 0\end{array}\right)$. Wolfram Alpha row reduces this to $\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right)$. So the only solution to the homogeneous linear system is $(x, y, z) = (0,0,0)$. In other words, only the trivial linear combination will give the zero vector as a result. That is, $x\begin{pmatrix}4\\5\\3\end{pmatrix} + y\begin{pmatrix}2\\2\\1\end{pmatrix} + z\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ is only true when x, y, z are all zero. **Conclude that the three vectors are linearly independent.**

(B) Observe $\begin{pmatrix}4\\2\\1\end{pmatrix} + \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}6\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ **The three vectors are linearly dependent.**

(C) Notice that the vectors in the two-element set $\left\{\begin{pmatrix}5\\4\\2\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right\}$ are NOT scalar multiples of one another. This tells us that the set is linearly independent. We can still go through the process of setting up an augmented matrix and performing row operations, just to confirm that we do get the same result. The augmented matrix is $\left(\begin{array}{cc|c}5 & 3 & 0\\ 4 & 2 & 0\\ 2 & 1 & 0\end{array}\right)$. Wolfram Alpha row reduces this to $\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$. So the only solution to the homogeneous system is $(x, y) = (0,0)$. In other words, only the trivial linear combination will give the zero vector as a result.That is, $x\begin{pmatrix}5\\4\\2\end{pmatrix} + y\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ is only true when x, y are both zero. **So the two vectors are linearly independent.**

(D) Notice that the collection $\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\6\end{pmatrix}$ of four vectors consists of the three vectors from question (A), and an additional fourth vector. We go through the same steps that we did in A,B,C. As above, we are led to an augmented matrix, $\left(\begin{array}{cccc|c}2 & 5 & 1 & 2 & 0\\ 2 & 4 & -1 & 5 & 0\\ 1 & 3 & 1 & 6 & 0\end{array}\right)$. Wolfram Alpha row reduces this to $\left(\begin{array}{cccc|c}1 & 0 & 0 & -50 & 0\\ 0 & 1 & 0 & 23 & 0\\ 0 & 0 & 1 & -13 & 0\end{array}\right)$.
We see that the homogeneous system has w as a free variable. This tells us that there will be many solutions to the homogeneous system. Therefore, there will exist many non-trivial linear combinations of $\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\6\end{pmatrix}$ that equal $\begin{pmatrix}0\\0\\0\end{pmatrix}$. But we must be clearer, and give an actual example of such a non-trivial linear combination. The reduced echelon augmented matrix tells us that w can be anything, but we must have $z = 13w$ and $y = -23w$ and $x = 50w$. That is, $50w\begin{pmatrix}2\\2\\1\end{pmatrix} - 23w\begin{pmatrix}5\\4\\3\end{pmatrix} + 13w\begin{pmatrix}1\\-1\\1\end{pmatrix} + w\begin{pmatrix}2\\5\\6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ for all $w \in \mathbb{R}$. We still must be even more explicit and give an example of a non-trivial linear combination.
Let $w = 1$ to get a non-trivial linear combination that works: $50\begin{pmatrix}2\\2\\1\end{pmatrix} - 23\begin{pmatrix}5\\4\\3\end{pmatrix} + 13\begin{pmatrix}1\\-1\\1\end{pmatrix} + \begin{pmatrix}2\\5\\6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

So the four vectors are linearly dependent.

[2] (A) We must determine if there exists a non-trivial linear combination of $2-x-x^2, 2+x, 7+5x+x^2$ that equals the 0 function. That is, we want to determine if there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that

$$c_1(2-x-x^2)+c_2(2+x)+c_3(7+5x+x^2)=0$$

Group coefficients of powers of x, the equation becomes

$$(2c_1+2c_2+7c_3)\cdot 1+(-c_1+c_2+5c_3)\cdot x+(-c_1+0c_2+c_3)\cdot x^2=0$$

This equation can only be true if the coefficients of the various powers of x on the left and right sides match.

So, we want to know if there is a non-trivial solution to homogeous system $\begin{cases} 2c_1 + 2c_2 + 7c_3 = 0 \\ -c_1 + c_2 + 5c_3 = 0 \\ -c_1 + 0c_2 + 1c_3 = 0 \end{cases}$

This system is represented by augmented matrix $\left(\begin{array}{ccc|c} 2 & 2 & 7 & 0 \\ -1 & 1 & 5 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right)$. Wolfram reduces this to $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$. This tells us that the only solution to the homogeneous linear system is $(c_1, c_2, c_3)=(0,0,0)$. In other words, only the trivial linear combination will give the zero function as a result. That is $c_1(2-x-x^2)+c_2(2+x)+c_3(7+5x+x^2)=0$ is only true when c_1, c_2, c_3 are all zero. **Conclude that the three functions are linearly independent.**

(B) We must determine if there exists a non-trivial linear combination of $1+2x^2, 6-3x+9x^2, 3-x+5x^2$ that equals the 0 function. That is, we want to determine if there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that

$$c_1(1+2x^2)+c_2(6-3x+9x^2)+c_3(3-x+5x^2)=0$$

As above, we are led to an augmented matrix, this time $\left(\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 0 & -3 & -1 & 0 \\ 2 & 9 & 5 & 0 \end{array}\right)$. This reduces to $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ (using Wolfram). We see that the homogeneous system has z as a free variable.This tells us that there will be many solutions to the homogeneous system. Therefore, there will exist many non-trivial linear combinations of $1+2x^2, 6-3x+9x^2, 3-x+5x^2$ that equal the zero function.

But we must be clearer, and give an actual example of such a non-trivial linear combination.

The reduced echelon matrix tells us that c_3 can be anything, but we must have $c_2=-\frac{1}{3}c_3$ and $c_1=-c_3$.

That is, $-c_3(1+2x^2)-\frac{1}{3}c_3(6-3x+9x^2)+c_3(3-x+5x^2)=0$ for all $c_3 \in \mathbb{R}$.

We still must be even more explicit and give an example of a non-trivial linear combination that works.

Let $c_3=-3$ to get an example of a non-trivial linear combination that works:

$$3\cdot(1+2x^2)+1\cdot(6-3x+9x^2)-3\cdot(3-x+5x^2)=0$$

Conclude that the three functions are linearly dependent.

[3] (a) Show that any set of four vectors in $\mathbb{R}^3$ is linearly dependent.

Solution: We mimic the textbook's solution of suggested exercise Two.II.1#34.

Suppose $\overrightarrow{v_1}=\begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{2,1} \end{pmatrix}$ and $\overrightarrow{v_2}=\begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{2,2} \end{pmatrix}$ and $\overrightarrow{v_3}=\begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{2,3} \end{pmatrix}$ and $\overrightarrow{v_4}=\begin{pmatrix} a_{1,4} \\ a_{2,4} \\ a_{2,4} \end{pmatrix}$ are any four vectors in $\mathbb{R}^3$.

We must determine if there exists a non-trivial linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}$ that equals $\vec{0}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

That is, we want to determine if there exist coefficients $c_1, c_2, c_3, c_4 \in \mathbb{R}$, not all zero, such that

$$c_1\cdot\overrightarrow{v_1}+c_2\cdot\overrightarrow{v_2}+c_3\cdot\overrightarrow{v_3}+c_4\cdot\overrightarrow{v_4}=\vec{0}$$

That is, we want to determine if there exist coefficients $c_1, c_2, c_3, c_4 \in \mathbb{R}$, not all zero, such that

$$c_1\cdot\begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{2,1} \end{pmatrix}+c_2\cdot\begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{2,2} \end{pmatrix}+c_3\cdot\begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{2,3} \end{pmatrix}+c_4\cdot\begin{pmatrix} a_{1,4} \\ a_{2,4} \\ a_{2,4} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

That is, we want to know if there is a solution to the homogeous linear system

$$\begin{cases} c_1a_{1,1}+c_2a_{1,2}+c_3a_{1,3}+c_4a_{1,4}=0 \\ c_1a_{2,1}+c_2a_{2,2}+c_3a_{2,3}+c_4a_{2,4}=0 \\ c_1a_{3,1}+c_2a_{3,2}+c_3a_{3,3}+c_4a_{4,4}=0 \end{cases}$$

This homogeneous linear system will definitely have a solution. (Remember that every homogeneous system has at least the zero vector as a solution. That is called the trivial solution.) But in fact we know that row operations will yield a reduced echelon form with at least one free variable (because there are more variables than equations, more columns than rows). So in fact, there will be many solutions besides the trivial solution So we know that there exist coefficients $c_1, c_2, c_3, c_4 \in \mathbb{R}$, not all zero, such that

$$c_1 \cdot \overrightarrow{v_1} + c_2 \cdot \overrightarrow{v_2} + c_3 \cdot \overrightarrow{v_3} + c_4 \cdot \overrightarrow{v_4} = \vec{0}$$

We conclude that the collection $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}$ of four vectors in $\mathbb{R}^3$ is linearly dependent.

[3] (b) What is the greatest number of elements that a linearly independent subset of $\mathbb{R}^3$ can have? Explain.
Solution: We know that it is possible for a linearly independent subset of $\mathbb{R}^3$ to have three elements, because the subset $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \subset \mathbb{R}^3$ is linearly independent. From part (a) of this question, we know that a linearly independent subset of $\mathbb{R}^3$ cannot have exactly four elements. What about subsets with more than four elements? Suppose that a subset $S \subset \mathbb{R}^3$ has more than four elements. Then S contains some subset $F \subset S$ that has exactly four elements. The subset F is linearly dependent because of part (a) of this question. Lemma 1.19 on p. 109 then tells us that S must also be linearly dependent because it is a superset of set F.
Conclude that the greatest number of elements that a linearly independent subset of $\mathbb{R}^3$ can have is three.

[4] (a) We must decide if $\left\langle \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle$ is a basis for$\mathbb{R}^3$. **Solution:** In [1](a), we found that the three vectors are linearly independent. That is part of the requirement for a basis. But we must also check to see if they span the space. That means that we must determine which vectors $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ can be obtained as a linear combination of $\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}$. That is, we want to determine if there exist $x, y, z \in \mathbb{R}$, not all zero, such tha $x\begin{pmatrix}2\\2\\1\end{pmatrix} + y\begin{pmatrix}5\\4\\3\end{pmatrix} + z\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$. That is, we want to know if there is a solution to the linear system

$$\begin{cases} 2x & + & 5y & + & 1z & = & a \\ 2x & + & 4y & - & 1z & = & b \\ 1x & + & 3y & + & 1z & = & c \end{cases}$$

This system is represented by the matrix $\left(\begin{array}{ccc|c} 2 & 5 & 1 & a \\ 2 & 4 & -1 & b \\ 1 & 3 & 1 & c \end{array}\right)$. Wolfram reduces this to $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7a-2b-9c \\ 0 & 1 & 0 & -3a+b+4c \\ 0 & 0 & 1 & 2a-b-2c \end{array}\right)$

This tells us that any vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ can be expressed as a linear combination using the coefficients

$$x = 7a - 2b - 9c$$
$$y = -3a + b + 4c$$
$$z = 2a - b - 2c$$

That is, $(7a-2b-9c)\begin{pmatrix}2\\2\\1\end{pmatrix} + (-3a+b+4c)\begin{pmatrix}5\\4\\3\end{pmatrix} + (2a-b-2c)\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$.

So the set of three vectors does span $\mathbb{R}^3$.
Since the set of three vectors is linearly independent and spans $\mathbb{R}^3$, we conclude that it is a basis for $\mathbb{R}^3$.

[4] (b) We must decide if $\left\langle\begin{pmatrix}4\\2\\1\end{pmatrix},\begin{pmatrix}2\\3\\1\end{pmatrix},\begin{pmatrix}6\\5\\2\end{pmatrix}\right\rangle$ is a basis for $\mathbb{R}^3$.

Solution: In [1](b), we found that the three vectors are linearly dependent. **So they cannot form a basis.**

[4] (c) We must decide if $\left\langle\begin{pmatrix}5\\4\\2\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\right\rangle$ is a basis for $\mathbb{R}^3$. **Solution:** In [1](c), we found that the two vectors are linearly independent. That is part of the requirement for a basis. But we must also check to see if they span the space. That means that we must determine which vectors $\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3$ can be obtained as a linear combination of $\begin{pmatrix}5\\4\\2\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}$. That is, we want to find if there exist $x,y,z\in\mathbb{R}$, not all zero, such tha $x\begin{pmatrix}5\\4\\2\end{pmatrix}+y\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$.

That is, we want to know if there is a solution to the system $\begin{cases}5x & + & 3y & = & a\\4x & + & 2y & = & b\\2x & + & 1y & = & c\end{cases}$ This linear system is represented by the matrix $\left(\begin{array}{cc|c}5 & 3 & a\\4 & 2 & b\\2 & 1 & c\end{array}\right)$. This row reduces to $\left(\begin{array}{cc|c}1 & 0 & -a-3b+c\\0 & 1 & 2a+2b-c\\0 & 0 & b-2c\end{array}\right)$ (done by hand).

Observe that the bottom row will represent an inconsistent equation unless $b-2c=0$. So the only vectors $\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3$ that can be expressed as a linear combination of $\begin{pmatrix}5\\4\\2\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}$ are those vectors with $b=2c$. Observe that the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}\in\mathbb{R}^3$ does not have that property. So the $\begin{pmatrix}1\\1\\1\end{pmatrix}\in\mathbb{R}^3$ is not in the span $\left[\left\{\begin{pmatrix}5\\4\\2\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\right\}\right]$.

Since the span is not all of $\mathbb{R}^3$, we conclude that even though the set of two vectors is linearly independent, it is not a basis for $\mathbb{R}^3$.

[4] (d) We must decide if $\left\langle\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}5\\4\\3\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}2\\5\\6\end{pmatrix}\right\rangle$ is a basis for $\mathbb{R}^3$. **Solution:** In [1](d), we found that the three vectors are linearly dependent. **So they cannot form a basis.**

[5] Using basis $D=\langle x^3+x^2+x+1, x^2+x+1, x+1, 1\rangle$ for $\mathcal{P}_3$, represent vector $ax^3+bx^2+cx+d\in\mathcal{P}_3$ with respect to the basis D. Show all details clearly.

Solution: We seek constants $c_1,c_2,c_3,c_4\in\mathbb{R}$ such that

$$c_1(x^3+x^2+x+1)+c_2(x^2+x+1)+c_3(x+1)+c_4(1)=ax^3+bx^2+cx+d$$

Grouping coefficients of powers of x on the left side, the equation becomes

$$(c_1)x^3+(c_1+c_2)x^2+(c_1+c_2+c_3)x+(c_1+c_2+c_3+c_4)(1)=ax^3+bx^2+cx+d$$

The coefficients of each power of x must match for this equation to be true. This gives us a system of equations:

$$\begin{cases}c_1=a\\c_1+c_2=b\\c_1+c_2+c_3=c\\c_1+c_2+c_3+c_4=d\end{cases} \Rightarrow \left(\begin{array}{cccc|c}1&0&0&0&a\\1&1&0&0&b\\1&1&1&0&c\\1&1&1&1&d\end{array}\right) \Rightarrow \text{operations} \Rightarrow \left(\begin{array}{cccc|c}1&0&0&0&a\\0&1&0&0&b-a\\0&0&1&0&c-b\\0&0&0&1&d-c\end{array}\right) \Rightarrow \begin{cases}c_1=a\\c_2=b-a\\c_3=c-b\\c_4=d-c\end{cases}.$$

Thus, $ax^3+bx^2+cx+d=(a)(x^3+x^2+x+1)+(b-a)(x^2+x+1)+(c-b)(x+1)+(d-c)(1)$

Now that we have the coefficients, we use them to build the representation. The representation is

$$\boldsymbol{Rep_D(ax^3+bx^2+cx+d)=\begin{pmatrix}a\\b-a\\c-b\\d-c\end{pmatrix}}$$