[1] Define map $f: \mathcal{P}_2 \to \mathbb{R}^3$ by $f(a+bx+cx^2) = \begin{pmatrix} 2c \\ b-a \\ c-b \end{pmatrix}$. The book would write $a+bx+cx^2 \xrightarrow{f} \begin{pmatrix} 2c \\ b-a \\ c-b \end{pmatrix}$.
Find the image of each of these elements of the domain: (a) $\overrightarrow{v_1} = 4-3x+2x^2$ (b) $\overrightarrow{v_2} = x+x^2$

Solution: (a) The image of $\overrightarrow{v_1}$ is the vector $f(\overrightarrow{v_1}) = f(4-3x+2x^2) = \begin{pmatrix} 2(2) \\ (-3)-4 \\ 2-(-3) \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 5 \end{pmatrix}$.

(b) The image of $\overrightarrow{v_2}$ is the vector $f(\overrightarrow{v_2}) = f(x+x^2) = \begin{pmatrix} 2(1) \\ 1-0 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

[2] Consider the isomorphim $Rep_\beta: \mathcal{P}_2 \to \mathbb{R}^3$, where β is the basis $\beta = \langle \overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3} \rangle = \langle 1, 1+x, 1+x+x^2 \rangle$ for $\mathcal{P}_2$. Find the image of each of these elements of the domain: (a) $\overrightarrow{v_1} = 7-5x+3x^2$ (b) $\overrightarrow{v_2} = x+x^2$.
Solution:

First note that the representation map $Rep_\beta: \mathcal{P}_2 \to \mathbb{R}^3$ works in the following way:

$$Rep_\beta(c_1 \cdot \overrightarrow{w_1} + c_2 \cdot \overrightarrow{w_2} + c_3 \cdot \overrightarrow{w_3}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

In order to determine the output, the input vector must be expressed as a linear combination of the basis vectors $\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}$ from basis β. So we start by expressing $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ as linear combinations of $\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}$.

(a) $\overrightarrow{v_1} = 7-5x+3x^2 = (12)(1) + (-8)(1+x) + 3(1+x+x^2) = 12\overrightarrow{w_1} - 8\overrightarrow{w_2} + 3\overrightarrow{w_3}$
(b) $\overrightarrow{v_2} = x+x^2 = (-1)(1) + (0)(1+x) + 1(1+x+x^2) = (-1)\overrightarrow{w_1} + (0)\overrightarrow{w_2} + (1)\overrightarrow{w_3}$

Now that we know those linear combinations, we can compute the representations.

(a) $Rep_\beta(\overrightarrow{v_1}) = Rep_\beta(5\overrightarrow{w_1} - 7\overrightarrow{w_2} + 4\overrightarrow{w_3}) = \begin{pmatrix} 12 \\ -8 \\ 3 \end{pmatrix}$

(b) $Rep_\beta(\overrightarrow{v_2}) = Rep_\beta\big((-1)\overrightarrow{w_1} + (0)\overrightarrow{w_2} + (1)\overrightarrow{w_3}\big) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

[3] Decide whether each map f is an isomorphism. If it is an isomorphism, then prove it. If it is not an isomorphism, then state a condition that it fails to satisfy.

(a) $f: \mathcal{M}_{2\times 2} \to \mathbb{R}$ defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = bc$.

Solution: This map is not an isomorphism. (Note that showing any one of these failures would be sufficient.)

***f* is not one-to-one.** To see why, let $\overrightarrow{v_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Then observe that $\overrightarrow{v_1} \neq \overrightarrow{v_2}$ and yet $f(\overrightarrow{v_1}) = f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 \cdot 1 = 0 = 1 \cdot 0 = f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f(\overrightarrow{v_2})$.

***f* also does not preserve vector addition.** To see why, let $\overrightarrow{v_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Then observe that $f(\overrightarrow{v_1} + \overrightarrow{v_2}) = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 1 \cdot 1 = 1$

Then observe that $f(\overrightarrow{v_1}) + f(\overrightarrow{v_2}) = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 1 \cdot 0 + 0 \cdot 1 = 0 + 0 = 0.$

So $f(\overrightarrow{v_1} + \overrightarrow{v_2}) \neq f(\overrightarrow{v_1}) + f(\overrightarrow{v_2})$.

f also does not preserve scalar multiplication. To see why, let $\vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $c = 2$.

Then observe that $f(c\vec{v}) = f\left(2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right) = 2 \cdot 2 = 4$

Then observe that $cf(\vec{v}) = 2f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 2 \cdot (1 \cdot 1) = 2.$

So $f(c\vec{v}) \neq cf(\vec{v})$.

(b) $f: \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b-a \\ c-b \\ d-c \end{pmatrix}$.

Solution: This map is an isomorphism.

f is one-to-one.

To see why, suppose $\overrightarrow{v_1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and suppose that $f(\overrightarrow{v_1}) = f(\overrightarrow{v_2})$.

Then $\begin{pmatrix} a_1 \\ b_1 - a_1 \\ c_1 - b_1 \\ d_1 - c_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 - a_2 \\ c_2 - b_2 \\ d_2 - c_2 \end{pmatrix}$.

The top entries must match, so $a_1 = a_2$.

Replacing a_1 with a_2 in the second entry and cancelling yields $b_1 = b_2$.

Similary, we can show that $c_1 = c_2$ and $d_1 = d_2$.

Conclude that $\overrightarrow{v_1} = \overrightarrow{v_2}$.

f is onto. To see why, suppose that $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ is any desired output in $\mathbb{R}^4$.

Then let $\vec{x} = \begin{pmatrix} y_1 & (y_2 + y_1) \\ (y_3 + y_2 + y_1) & (y_4 + y_3 + y_2 + y_1) \end{pmatrix}$. Observe that

$$f(\vec{x}) = f\begin{pmatrix} y_1 & (y_2 + y_1) \\ (y_3 + y_2 + y_1) & (y_4 + y_3 + y_2 + y_1) \end{pmatrix} = \begin{pmatrix} y_1 \\ (y_2 + y_1) - y_1 \\ (y_3 + y_2 + y_1) - (y_2 + y_1) \\ (y_4 + y_3 + y_2 + y_1) - (y_3 + y_2 + y_1) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \vec{y}$$

We have found an $\vec{x}$ such that $f(\vec{x}) = \vec{y}$.

f preserves vector addition. To see why, suppose $\overrightarrow{v_1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then

$$f(\overrightarrow{v_1} + \overrightarrow{v_2}) = f\begin{pmatrix} (a_1 + a_2) & (b_1 + b_2) \\ (c_1 + c_2) & (d_1 + d_2) \end{pmatrix} = \begin{pmatrix} (a_1 + a_2) \\ (b_1 + b_2) - (a_1 + a_2) \\ (c_1 + c_2) - (b_1 + b_2) \\ (d_1 + d_2) - (c_1 + c_2) \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 - a_1 \\ c_1 - b_1 \\ d_1 - c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 - a_2 \\ c_2 - b_2 \\ d_2 - c_2 \end{pmatrix}$$

$$= f\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + f\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = f(\overrightarrow{v_1}) + f(\overrightarrow{v_2})$$

f preserves scalar multiplication. To see why, suppose $\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $r \in \mathbb{R}$. Then

$$f(r\vec{v}) = f\left(r\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = \begin{pmatrix} ra \\ rb - ra \\ rc - rb \\ rd - rc \end{pmatrix} = r\begin{pmatrix} ra \\ r(b-a) \\ r(c-b) \\ r(d-c) \end{pmatrix} = r\begin{pmatrix} a \\ b-a \\ c-b \\ d-c \end{pmatrix} = rf(\vec{v})$$

(c) $f: \mathcal{M}_{2\times2} \to \mathcal{P}_3$ defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 + (a+b)x + (b+c)x^2 + (c+d)x^3$.

Solution: This map is not an ismorphism. To see why, observe that $f(\vec{0}) = f\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 1 \neq \vec{0}$, so we know that f cannot preserve vector space operations.

[4] (a) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is not an isomorphism. Why not? (Explain which of the isomorphism requirements the function fails.)

Solution: (Any one of these failures would be sufficient to prove that *f* is not an isomorphism.)

***f* does not preserve vector addition.** To see why, let $\vec{v_1} = 1$ and $\vec{v_2} = 1$.
Then observe that $f(\vec{v_1} + \vec{v_2}) = f(1+1) = f(2) = 8$
and observe that $f(\vec{v_1}) + f(\vec{v_2}) = f(1) + f(1) = 1 + 1 = 2$.
So $f(\vec{v_1} + \vec{v_2}) \neq f(\vec{v_1}) + f(\vec{v_2})$.

***f* aso does not preserve scalar multiplication.** To see why, let $\vec{v} = 1$ and $c = 2$.
Then observe that $f(c\vec{v}) = f(2 \cdot 1) = f(2) = 8$
Then observe that $cf(\vec{v}) = 2f(1) = 2 \cdot 1 = 2$.
So $f(c\vec{v}) \neq cf(\vec{v})$.

(b) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is not an isomorphism. Why not? (Explain which of the isomorphism requirements the function fails.)

Solution: (Any one of these failures would be sufficient to prove that *f* is not an isomorphism.)

***f* is not onto.** To see why, let $y = -1$. There is no x such that $f(x) = y$.

***f* does not preserve vector addition.** To see why, let $\vec{v_1} = 0$ and $\vec{v_2} = 0$.
Then observe that $f(\vec{v_1} + \vec{v_2}) = f(0+0) = f(0) = e^0 = 1$
and observe that $f(\vec{v_1}) + f(\vec{v_2}) = f(0) + f(0) = e^0 + e^0 = 1 + 1 = 2$.
So $f(\vec{v_1} + \vec{v_2}) \neq f(\vec{v_1}) + f(\vec{v_2})$.

***f* also does not preserve scalar multiplication**. To see why, let $\vec{v} = 0$ and $c = 2$.
Then observe that $f(c\vec{v}) = f(2 \cdot 0) = f(0) = e^0 = 1$
Then observe that $cf(\vec{v}) = 2f(0) = 2 \cdot e^0 = 2 \cdot 1 = 2$.
So $f(c\vec{v}) \neq cf(\vec{v})$.

(c) Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one. Your function must be unique, not the same function as anybody else in the class.

Solution:

We discussed one example in class: the cubic function $f(x) = (x-a)(x-b)(x-c)$ with a, b, c not identical would be onto but not one-to-one. Observe that because it is a cubic polynomial with positive leading coefficient, its graph goes down on the left and up on the right (so it is ONTO), and the graph has x intercepts at $x = a$ and $x = b$ and $x = c$ (so it is not one-to-one). Choose any three numbers a, b, c not identical to build an actual example of such a function. My example is unique in all the world:

$$f(x) = (x-13)(x-17)(x-19)$$

(d) Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that is an isomorphism. Again, your function must be unique.
Solution:

As we discussed in class, for a function $f: \mathbb{R} \to \mathbb{R}$ to preserve vector space operations (vector addition and scalar multiplication of vectors) the function will have to be of the form $f(x) = kx$ for some constant k. (You will prove this in problem [5].) The additional requirement that f be one-to-one means that the constant k must be non-zero (see my proof below). Choose any constant $k \neq 0$ to build an actual example of such a function $f(x) = kx$. My example is unique in all the world:

$$f(x) = 54.17385x$$

[5] Prove that if $f: \mathbb{R} \to \mathbb{R}$ is an isomorphism then f must be of the form $f(x) = kx$ where k is some non-zero real number. That is, prove that if $f: \mathbb{R} \to \mathbb{R}$ is an isomorphism then there exists a non-zero real number k such that $f(x) = kx$.

Proof:(The first part of this proof was done in class.)

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is an isomorphism.
Then f passes the four isomorphism tests.

Use the fact that f passes test 2

In particular, f passes test 2. That is, f preserves scalar multiplication.
That means that the equation $f(rx) = rf(x)$ is true for all real numbers r and x.
In particular, the equation, is true when $x = 1$.
So the equation $f(r) = rf(1)$ is true for all real numbers r.
We can switch the order on the right side: The equation $f(r) = f(1)r$ is true for all real numbers r.
And we can change the letter: The equation $f(t) = f(1)t$ is true for all real numbers t.
We can even use the letter x: The equation $f(x) = f(1)x$ is true for all real numbers x.
Let k be the real number $k = f(1)$.
Observe that we have found a real number k such that $f(x) = kx$ is true for all real numbers x.

Use the fact that f passes test 3 to prove that $k = f(1) \neq 0$.

So far, we have found our number k, but we have not proven that k is non-zero.
Now we must prove that k is non-zero. But k was defined as $k = f(1)$, so we must prove that $f(1) \neq 0$.
We will prove this by contradiction. That is, we will assume that $f(1) = 0$, and show that that leads to a contradiction.
Assume that $f(1) = 0$.
Then $f(2) = f(2 \cdot 1) = 2f(1) = 2(0) = 0$.
So even though $1 \neq 2$, it would turn out that $f(1) = f(2)$.
This would tell us that f is not one-to-one.
But that would contradict the fact that f is known to be one-to-one, because f passes test 3.
Therefore our assumption that $f(1) = 0$ was incorrect.
Conclude that $f(1)$ must not be zero.
We have proven that $k = f(1) \neq 0$.

Conclusion

We have found a non-zero real number k such that $f(x) = kx$ is true for all real numbers x.

End of Proof