[1] Consider the isomorphim $Rep_\beta : \mathcal{P}_2 \to \mathbb{R}^3$, where β is the basis $\beta = \langle \overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3} \rangle = \langle 1, 1+x, 1+x+x^2 \rangle$ for $\mathcal{P}_2$. Every isomorphism has an inverse. Find ${Rep_\beta}^{-1}\begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix}$.

Solution: First note that the inverse map ${Rep_\beta}^{-1}: \mathbb{R}^3 \to \mathcal{P}_2$ works as the reverse of the map Rep_β as follows: ${Rep_\beta}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \cdot \overrightarrow{w_1} + c_2 \cdot \overrightarrow{w_2} + c_3 \cdot \overrightarrow{w_3}$. So ${Rep_\beta}^{-1}\begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = 7\overrightarrow{w_1} + (-1)\overrightarrow{w_2} + 3\overrightarrow{w_3} = 7(1) - 5(1+x) + 3(1+x+x^2) = 5 - 2x + 3x^2$.

[2] Decide whether each map f is linear. (That is, decide if it is a homomorphism.) If it is linear, then prove it. If it is not an linear, then state a condition that it fails to satisfy. Bonus: Determine if each is an isomorphism.

(a) $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$. (This is an example of a "projection map".)

Solution:

Let $\overrightarrow{v_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $r_1, r_2 \in \mathbb{R}$. We must show that $f(r_1\overrightarrow{v_1} + r_2\overrightarrow{v_2}) = r_1 f(\overrightarrow{v_1}) + r_1 f(\overrightarrow{v_2})$.

$f(r_1\overrightarrow{v_1} + r_2\overrightarrow{v_2})$	$= f\left(r_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$	definition of $\overrightarrow{v_1}, \overrightarrow{v_2}$.
	$= f\left(\begin{pmatrix} r_1 a_1 \\ r_1 b_1 \end{pmatrix} + \begin{pmatrix} r_2 a_2 \\ r_2 b_2 \end{pmatrix}\right)$	definition of scalar multiplication
	$= f\begin{pmatrix} r_1 a_1 + r_2 a_2 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$	definition of vector addition
	$= \begin{pmatrix} 0 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$	definition of how f works.
	$= \begin{pmatrix} 0 \\ r_1 b_1 \end{pmatrix} + \begin{pmatrix} 0 \\ r_2 b_2 \end{pmatrix}$	definition of vector addition
	$= r_1\begin{pmatrix} 0 \\ b_1 \end{pmatrix} + r_2\begin{pmatrix} 0 \\ b_2 \end{pmatrix}$	definition of scalar multiplication
	$= r_1 f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$	definition of how f works.
	$= r_1 f(\overrightarrow{v_1}) + r_1 f(\overrightarrow{v_2})$	definition of $\overrightarrow{v_1}, \overrightarrow{v_2}$.

Conclude that f is linear.

Notice that f is not an isomorphism because:

It is not one-to-one: $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

It is not onto: Given the desired output $\vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, there is no input $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $f(\vec{v}) = \vec{y}$.

(b) $f: \mathbb{R}^2 \to \mathcal{P}_2$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = (a+b) + bx^2$.

Solution:

Let $\overrightarrow{v_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $r_1, r_2 \in \mathbb{R}$. We must show that $f(r_1\overrightarrow{v_1} + r_2\overrightarrow{v_2}) = r_1 f(\overrightarrow{v_1}) + r_1 f(\overrightarrow{v_2})$.

$f(r_1\overrightarrow{v_1} + r_2\overrightarrow{v_2})$	$= f\left(r_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$	definition of $\overrightarrow{v_1}, \overrightarrow{v_2}$.
	$= f\left(\begin{pmatrix} r_1 a_1 \\ r_1 b_1 \end{pmatrix} + \begin{pmatrix} r_2 a_2 \\ r_2 b_2 \end{pmatrix}\right)$	definition of scalar multiplication
	$= f\begin{pmatrix} r_1 a_1 + r_2 a_2 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$	definition of vector addition
	$= (r_1 a_1 + r_2 a_2 + r_1 b_1 + r_2 b_2) + (r_1 b_1 + r_2 b_2)x^2$	definition of how f works.
	$= r_1\big((a_1 + b_1) + b_1 x^2\big) + r_2\big((a_2 + b_2) + b_2 x^2\big)$	arithmetic

$= r_1 f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ definition of how f works.
$= r_1 f(\vec{v_1}) + r_1 f(\vec{v_2})$ definition of $\vec{v_1}, \vec{v_2}$.

Conclude that f is linear.

Notice that f is one-to-one: Suppose that $f(\vec{v_1}) = f(\vec{v_2})$ for some input vectors $\vec{v_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$. Then $f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, so $(a_1 + b_1) + b_1 x^2 = (a_2 + b_2) + b_2 x^2$. Equating coefficients of x^2 tells us that $b_1 = b_2$. Then equating constant terms tells us that $(a_1 + b_1) = (a_2 + b_2)$. Substituting in $b_1 = b_2$, tells us that $a_1 = a_2$. Therefore, $\vec{v_1} = \vec{v_2}$.

But f is not onto: Given the desired output $\vec{y} = x$, there is no input $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $f(\vec{v}) = \vec{y}$.

Conclude that f is not an isomorphism.

(c) $f: \mathbb{R}^2 \to \mathcal{P}_1$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = ax + b$.

Solution:

Let $\vec{v_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $r_1, r_2 \in \mathbb{R}$. We must show that $f(r_1\vec{v_1} + r_2\vec{v_2}) = r_1 f(\vec{v_1}) + r_1 f(\vec{v_2})$.

$f(r_1\vec{v_1} + r_2\vec{v_2}) = f\left(r_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of $\vec{v_1}, \vec{v_2}$.
$= f\begin{pmatrix} r_1 a_1 + r_2 a_2 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$ definition of scalar multiplication & vector addition
$= (r_1 a_1 + r_2 a_2)x + (r_1 b_1 + r_2 b_2)$ definition of how f works.
$= r_1(a_1 x + b_1) + r_2(a_2 x + b_2)$ arithmetic
$= r_1 f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ definition of how f works.
$= r_1 f(\vec{v_1}) + r_1 f(\vec{v_2})$ definition of $\vec{v_1}, \vec{v_2}$.

Conclude that f is linear.

Notice that f is one-to-one: Suppose that $f(\vec{v_1}) = f(\vec{v_2})$ for some input vectors $\vec{v_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$. Then $f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, so $a_1 x + b_1 = a_2 x + b_2$. Equating coefficients of powers of x tells us that $a_1 = a_2$ and $b_1 = b_2$. Therefore, $\vec{v_1} = \vec{v_2}$. **And f is onto:** Given any desired output $\vec{y} = ax + b$, the input $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ will give that desired output. That is, $f(\vec{v}) = \vec{y}$.

Conclude that f is isomorphism.

[3] Stating that a function is linear is different from stating that its graph is a line. The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 7$ has a graph that is a line. Show that it is not a linear function

Solution:

Show that f does not preserve vector addition. Let $\vec{v_1} = 1 \in \mathbb{R}$ and $\vec{v_2} = 1 \in \mathbb{R}$.
Then observe that $f(\vec{v_1} + \vec{v_2}) = f(1 + 1) = f(2) = 5(2) + 7 = 17$
Then observe that $f(\vec{v_1}) + f(\vec{v_2}) = f(1) + f(1) = (5(1) + 72) + (5(1) + 7) = 12 + 12 = 24$.
So $f(\vec{v_1} + \vec{v_2}) \neq f(\vec{v_1}) + f(\vec{v_2})$.

Show that f also does not preserve scalar multiplication. Let $\vec{v} = 1$ and $c = 2$.
Then observe that $f(c\vec{v}) = f(2(1)) = f(2) = 5(2) + 7 = 17$
Then observe that $cf(\vec{v}) = 2f(1) = 2 \cdot (5(1) + 7) = 2 \cdot (12) = 24$.
So $f(c\vec{v}) \neq cf(\vec{v})$.

[4] (a) The map the "evaluation at 5 map". That is the map $eval_5: \mathcal{F} \to \mathbb{R}$ defined by $eval_5(f) = f(5)$.

Solution:

Let $\vec{v_1} = f$ and $\vec{v_2} = g$ and $a, b \in \mathbb{R}$. We must show that $eval_5(af + bg) = a \cdot eval_5(f) + b \cdot eval_5(g)$.

$eval_5(af+bg) = (af+bg)(5)$ definition of how $eval_5$ works
$= (af)(5) + (bg)(5)$ definition of function addition
$= a \cdot f(5) + b \cdot g(5)$ definition of scalar multiplication of functions
$= a \cdot eval_5(f) + b \cdot eval_5(g)$ definition of how $eval_5$ works

Conclude that f is linear.

Notice that $eval_5$ is not one-to-one:
Let $f(x) = x^2$ and let $g(x) = x + 20$. Then $f \neq g$, but $eval_5(f) = f(5) = (5)^2 = 25$ and $eval_5(g) = g(5) = 20 + (5) = 25$. So $eval_5(f) = eval_5(g)$ even though $f \neq g$.

Notice that $eval_5$ is onto:
Given the desired output $\vec{y} = r \in \mathbb{R}$, let f be the constant function $f(x) = r$. Then $eval_5 f = f(5) = r$.

Conclude that since $eval_5$ is not one-to-one, it is not an isomorphism.

(b) The map "Definite Integral from 0 to 1". That is, the map $I: C^0 \to \mathbb{R}$ defined by $I(f) = \int_{t=0}^{t=1} f(t)dt$.

Solution: Show that the map *I* preserves all linear combinations.

Suppose $\overrightarrow{v_1} = f \in C^0$ and $\overrightarrow{v_2} = g \in C^0$ and $a, b \in \mathbb{R}$. Then

$I(a \cdot \overrightarrow{v_1} + b \cdot \overrightarrow{v_2}) = I(a \cdot f + b \cdot g)$ The vectors are the functions f, g.

$= \int_{t=0}^{t=1} (af+bg)(t)dt$ by the definition of I

$= \int_{t=0}^{t=1} \big((af)(t) + (bg)(t)\big)dt$ by the definition of function addition

$= \int_{t=0}^{t=1} \big(af(t) + bg(t)\big)dt$ by the definition of scalar multiplication

$= a \cdot \left(\int_{t=0}^{t=1} f(t)dt\right) + b \cdot \left(\int_{t=0}^{t=1} g(t)dt\right)$ by the *Sum Rule and Constant Multiple Rule for Integrals* From Calculus

$= a \cdot I(f) + b \cdot I(g)$ by the definition of I

$= aI(\overrightarrow{v_1}) + bI(\overrightarrow{v_2})$ The vectors are the functions f, g.

Conclude that *I* is linear. That is, *I* is a homomorphism.

Observe that *I* is not one-to-one

Let $\overrightarrow{v_1} = f$ be the function $f(x) = 1$ and Let $\overrightarrow{v_2} = g$ be the function $g(x) = 2x$. Then $\overrightarrow{v_1} \neq \overrightarrow{v_2}$.
But $I(\overrightarrow{v_1}) = I(f) = \int_{t=0}^{t=1} 1dt = 1$ while $I(\overrightarrow{v_2}) = I(g) = \int_{t=0}^{t=1} 2tdt = 1$, so $I(\overrightarrow{v_1}) = I(\overrightarrow{v_2})$.

Observe that *I* is onto

Let $y \in \mathbb{R}$ be any desired output.
Define the input vector $\vec{v} = f$ to be the constant function $f(x) = y$.
Then $I(\vec{v}) = I(f) = \int_{t=0}^{t=1} ydt = y \cdot \int_{t=0}^{t=1} dt = y \cdot 1 = y$.
We have found an input that gives the desired output.

Since I is not one-to-one, it is not an isomorphism.

[5] (a) The map "Definite Integral from 0 to x". That is, the map $I: C^0 \to C^1$ defined by $I(f) = \int_{t=0}^{t=x} f(t)dt$.

Solution:

The proof that the map I preserves all linear combinations is just like the proof in [4](b). The fact that the upper limit of integration is $t = x$ rather than $t = 1$ does not change any of the steps or any of the justifications.

Conclude that *I* is linear. That is, *I* is a homomorphism.

Observe that *I* is one-to-one

Suppose that $\overrightarrow{v_1} = f$ and $\overrightarrow{v_2} = g$ are two functions such that $I(f) = I(g)$.
As discussed in class, $I(f) = \int_{t=0}^{t=x} f(t)dt$ will be a function that is an antiderivative of $f(x)$, and $I(g)$ will be a function that is an antiderivative of $g(x)$.
So, if we take $\frac{d}{dx}$ of both sides of the equation $I(f) = I(g)$, we will obtain the new equation

$$\frac{d}{dx}I(f) = \frac{d}{dx}I(g)$$
$$f(x) = g(x)$$

Conclude that $f = g$, so $\overrightarrow{v_1} = \overrightarrow{v_2}$.

Observe that *I* is not onto

As discussed in class, $I(f) = \int_{t=0}^{t=x} f(t)dt$ will be a function of the variable x.
If we substitute $x = 0$ into this function, we obtain

$$\big(I(f)\big)(0) = \int_{t=0}^{t=0} f(t)dt = 0$$

because the lower and upper limits of integration are the same.
So $I(f) = \int_{t=0}^{t=x} f(t)dt$ will not only be a function that is an antiderivative of $f(x)$, it will be the particular antiderivative that has value 0 at $x = 0$. (In calculus, you sometimes denoted an antiderivative of f by F. So $I(f)$ is the particular antiderivative $F(x)$ that has the property that $F(0) = 0$.
So consider the desired output vector $\vec{y}$ that is the function $\cos(x)$. observe that this desired output function has the property that $\cos(0) = 1$, not 0. Therere, we know that there is no input vector $\vec{v}$, that is no input function f, such that $I(\vec{v}) = \vec{y}$.

Since *I* is not onto, it is not an isomorphism.

(b) The "Derivative" map That is, the map $D: C^1 \to C^0$ defined by $D(f) = \frac{df}{dx}$.

Solution:

Show that *D* preserves linear combinations. Suppose $\overrightarrow{v_1} = f \in C^1$ and $\overrightarrow{v_2} = g \in C^1$ and $a, b \in \mathbb{R}$. Then

$D(a\overrightarrow{v_1} + b\overrightarrow{v_2}) = D(af + bg)$	The vectors are the functions f, g.
$= \frac{d}{dx}\big((af + bg)(x)\big)$	by the definition of D
$= \frac{d}{dx}\big((af)(x) + (bg)(x)\big)$	definition of function addition
$= \frac{d}{dx}\big(a \cdot f(x) + b \cdot g(x)\big)$	definition of scalar multiplication
$= a\frac{df}{dx} + b\frac{dg}{dx}$	by the Sum Rule & Constant multiple rules for Derivatives
$= aD(f) + bD(g)$	by the definition of D
$= aD(\overrightarrow{v_1}) + bD(\overrightarrow{v_2})$	The vectors are the functions f, g.

Conclude that *D* is linear. That is, *D* is a homomorphism.

Notice that *D* is not one-to-one: Let $f(x) = x^2$ and let $g(x) = x^2 + 20$. Then $f \neq g$, but $D(f) = 2x$ and $D(g) = 2x$.So $D(f) = D(g)$ even though $f \neq g$.

Notice that *D* is onto:

Given the desired output $\vec{y} = f \in C^0$ let F be an antiderivative for f. (We know that such an antiderivative exists because f is continuous. In fact, we know that one way to get an antiderivative F is by defining $F = I(f)$, where I is The map "Definite Integral from 0 to x". That is, the map $I: C^0 \to C^1$ defined by $I(f) = \int_{t=0}^{t=x} f(t)dt$ from part (a) of this sproblem

Conclude that since *D* is not one-to-one, it is not an isomorphism.