[1] Define map $f: \mathcal{P}_2 \to \mathcal{P}_3$ by $f\big(p(x)\big) = x \cdot p(x)$. For example, $f(5 + 4x - x^2) = x \cdot (5 + 4x - x^2)$
Which of these vectors are in the range space of f? Explain. (If you think a vector is in the range space, you have to give an example of an input vector that will produce that vector as an output.)

(a) $\vec{v_a} = x^2$ (b) $\vec{v_b} = 2x + 13x^2$ (c) $\vec{v_c} = 2 + 13x^2$ (d) $\vec{v_d} = 0$ (e) $\vec{v_e} = 2$

Solution:

(a) Observe that $f(x) = x \cdot x = x^2 = \vec{v_a}$, so $\vec{v_a} = x^2$ is in the range space.
(b) Observe that $f(2 + 13x) = x \cdot (2 + 13x) = 2x + 13x^2 = \vec{v_b}$, so $\vec{v_b} = 2x + 13x^2$ is in the range.
(c) Observe that there is no polynomial $p(x)$ such that $f\big(p(x)\big) = x \cdot p(x) = 2 + 13x^2 = \vec{v_c}$.
So $\vec{v_c} = 2 + 13x^2$ is not in the range.
(d) Observe that $f(0) = x \cdot (0) = 0 = \vec{v_d}$, so $\vec{v_d} = 0$ is in the range.
(e) Observe that there is no polynomial $p(x)$ such that $f\big(p(x)\big) = x \cdot p(x) = 2 = \vec{v_e}$.
So $\vec{v_e} = 2$ is not in the range.

For each map in problems [2], [3], [4], answer the following:
(a) Is the map onto? (Explain)
(b) Find the range space. (Explain) (The definition of range space is on page 192.)
(c) Fnd a basis for the range space. (Explain)
(d) Find the rank of the map. (Explain) (The definition of range space is on page 192.)

[2] The map $f: \mathcal{P}_2 \to \mathcal{P}_3$ by $f\big(p(x)\big) = x \cdot p(x)$ that was introduced in problem [1].
(a) The map is not onto. We found, for example, that $\vec{v_c} = 2 + 13x^2$ is not in the range space
(b) The range space is $\mathcal{R}(f) = \{ax + bx^2 + cx^3 | a, b, c \in \mathbb{R}\}$
(c) A basis for the range space could be $\beta = \{x, x^2, x^3\}$.
(d) Since a basis for $\mathcal{R}(f)$ has three basis vectors, we conclude that $rank(f) = dim\big(\mathcal{R}(f)\big) = 3$.

[3] The differentiation map $D: \mathcal{P}_2 \to \mathcal{P}_2$ defined by $D(f) = \frac{d}{dx} f(x)$
(a) The map is not onto. Consider the desired output vector $\vec{y} = x^2$. There is no 2nd degree polynomial f such that $D(f) = \vec{y} = x^2$, because if f has degree 2 (or less), then $D(f)$ will have degree 1 (or less).
(b) The range space is $\mathcal{R}(D) = \{a + bx | a, b \in \mathbb{R}\} = \mathcal{P}_1$.
(c) A basis for the range space could be $\beta = \{1, x\}$.
(d) Since a basis for $\mathcal{R}(D)$ has three basis vectors, we conclude that $rank(D) = dim\big(\mathcal{R}(D)\big) = 2$.

[4] The map $f: \mathbb{R}^2 \to \mathcal{P}_2$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = 2bx - 5bx^2$.
(a) The map is not onto. Consider the desired output vector $\vec{y} = 1$. There is no input vector $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $f\begin{pmatrix} a \\ b \end{pmatrix} = 2bx - 5bx^2 = 1 = \vec{y}$.
(b) The range space is $\mathcal{R}(f) = \{b(2x - 5x^2) | b \in \mathbb{R}\}$.
(c) A basis for the range space could be $\beta = \{2x - 5x^2\}$.
(d) Since a basis for $\mathcal{R}(f)$ has one basis vector, we conclude that $rank(f) = dim\big(\mathcal{R}(f)\big) = 1$.

[5] The map $g: \mathcal{M}_{2\times 2} \to \mathbb{R}$ defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b + c + d$.
(a) The map is onto. Consider the desired output vector $\vec{y} = r \in \mathbb{R}$. Let the input vector be $\vec{x} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$.
Observe that $g(\vec{x}) = g\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = r + 0 + 0 + 0 = r = \vec{y}$.
(b) The range space is $\mathcal{R}(g) = \mathbb{R}$.
(c) A basis for the range space could be $\beta = \{1\}$.
(d) Since a basis for $\mathcal{R}(g)$ has one basis vector, we conclude that $rank(g) = dim\big(\mathcal{R}(g)\big) = 1$.