Let α, β, γ be the standard bases for $\mathbb{R}, \mathcal{P}_2, \mathcal{P}_3$. That is, let $\alpha = \langle 1 \rangle$ and $\beta = \langle 1, x, x^2 \rangle$ and $\gamma = \langle 1, x, x^2, x^3 \rangle$. And let δ be the non-standard basis $\delta = \langle 1, 2x, 3x^2 \rangle$ for $\mathcal{P}_2$.

[1] Let D be the map $D: \mathcal{P}_3 \to \mathcal{P}_2$ defined by $D(f) = f'$.

(a) Find $Rep_{\gamma,\beta}(D)$. **Solution:** We start by computing the outputs when the vectors of basis γ are used as input.

$$D(\overrightarrow{\gamma_1}) = D(1) = \frac{d}{dx}(1) = 0 = 0(1) + 0(x) + 0(x^2) = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}$$
$$D(\overrightarrow{\gamma_2}) = D(x) = \frac{d}{dx}(x) = 1 = 1(1) + 0(x) + 0(x^2) = 1\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}$$
$$D(\overrightarrow{\gamma_3}) = D(x^2) = \frac{d}{dx}(x^2) = 2x = 0(1) + 2(x) + 0(x^2) = 0\overrightarrow{\beta_1} + 2\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}$$
$$D(\overrightarrow{\gamma_4}) = D(x^3) = \frac{d}{dx}(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2) = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 3\overrightarrow{\beta_3}$$

Now we find the representations of these output vectors in the basis β.

$$Rep_\beta\left(D(\overrightarrow{\gamma_1})\right) = Rep_\beta\left(0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$Rep_\beta\left(D(\overrightarrow{\gamma_2})\right) = Rep_\beta\left(1\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$Rep_\beta\left(D(\overrightarrow{\gamma_3})\right) = Rep_\beta\left(0\overrightarrow{\beta_1} + 2\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}\right) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$
$$Rep_\beta\left(D(\overrightarrow{\gamma_4})\right) = Rep_\beta\left(0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 3\overrightarrow{\beta_3}\right) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Finally, we use those four column vectors to build a four-column matrix. $Rep_{\gamma,\beta}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) Find $Rep_{\gamma,\delta}(D)$. **Solution:** We start by computing the outputs when the vectors of basis γ are used as input.

$$D(\overrightarrow{\gamma_1}) = D(1) = \frac{d}{dx}(1) = 0 = 0(1) + 0(2x) + 0(3x^2) = 0\overrightarrow{\delta_1} + 0\overrightarrow{\delta_2} + 0\overrightarrow{\delta_3}$$
$$D(\overrightarrow{\gamma_2}) = D(x) = \frac{d}{dx}(x) = 1 = 1(1) + 0(2x) + 0(2x^2) = 1\overrightarrow{\delta_1} + 0\overrightarrow{\delta_2} + 0\overrightarrow{\delta_3}$$
$$D(\overrightarrow{\gamma_3}) = D(x^2) = \frac{d}{dx}(x^2) = 2x = 0(1) + 1(2x) + 0(3x^2) = 0\overrightarrow{\delta_1} + 1\overrightarrow{\delta_2} + 0\overrightarrow{\delta_3}$$
$$D(\overrightarrow{\gamma_4}) = D(x^3) = \frac{d}{dx}(x^3) = 3x^2 = 0(1) + 0(2x) + 1(3x^2) = 0\overrightarrow{\delta_1} + 0\overrightarrow{\delta_2} + 1\overrightarrow{\delta_3}$$

As in the previous problem we find the representations of these output vectors in the basis β. The results are

$$Rep_\delta\left(D(\overrightarrow{\gamma_1})\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } Rep_\delta\left(D(\overrightarrow{\gamma_2})\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } Rep_\delta\left(D(\overrightarrow{\gamma_3})\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } Rep_\delta\left(D(\overrightarrow{\gamma_4})\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Finally,Finally, we use those four column vectors to build a four-column matrix: $Rep_{\gamma,\delta}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

[2] Let $I_{[0,x]}$ be the map $I_{[0,x]}: \mathcal{P}_2 \to \mathcal{P}_3$ defined by $I_{[0,x]}(f) = \int_{t=0}^{t=x} f(t)dt$.

(a) Find $Rep_{\beta,\gamma}\left(I_{[0,x]}\right)$ **Solution:** We start by computing the outputs when the vectors of basis β are input.

$$I_{[0,x]}(\overrightarrow{\beta_1}) = I_{[0,x]}(1) = \int_{t=0}^{t=x} 1dt = x = 0(1) + 1(x) + 0(x^2) + 0(x^3) = 0\overrightarrow{\gamma_1} + 1\overrightarrow{\gamma_2} + 0\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$

$$I_{[0,x]}(\overrightarrow{\beta_2}) = I_{[0,x]}(x) = \int_{t=0}^{t=x} tdt = \frac{x^2}{2} = 0(1) + 0(x) + \frac{1}{2}(x^2) + 0(x^3) = 0\overrightarrow{\gamma_1} + 0\overrightarrow{\gamma_2} + \frac{1}{2}\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$

$$I_{[0,x]}(\overrightarrow{\beta_3}) = I_{[0,x]}(x^2) = \int_{t=0}^{t=x} t^2dt = \frac{x^3}{3} = 0(1) + 0(x) + 0(x^2) + \frac{1}{3}(x^3) = 0\overrightarrow{\gamma_1} + 0\overrightarrow{\gamma_2} + 0\overrightarrow{\gamma_3} + \frac{1}{3}\overrightarrow{\gamma_4}$$

As in the previous problem, we find the representations of these output vectors in the basis γ.

The results are $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\beta_1})\right) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\beta_2})\right) = \begin{pmatrix}0\\0\\1/2\\0\end{pmatrix}$ and $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\beta_3})\right) = \begin{pmatrix}0\\0\\0\\1/3\end{pmatrix}$.

Finally, we use those three column vectors to build a three-column matrix. $Rep_{\beta,\gamma}(I_{[0,x]}) = \begin{pmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & 1/3\end{pmatrix}$

(b) Find $Rep_{\delta,\gamma}(I_{[0,x]})$. **Solution:** We start by computing the outputs when the vectors of basis δ are input.

$$I_{[0,x]}(\overrightarrow{\delta_1}) = I_{[0,x]}(1) = \int_{t=0}^{t=x} 1dt = x = 0(1) + 1(x) + 0(x^2) + 0(x^3) = 0\overrightarrow{\gamma_1} + 1\overrightarrow{\gamma_2} + 0\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$

$$I_{[0,x]}(\overrightarrow{\delta_2}) = I_{[0,x]}(2x) = \int_{t=0}^{t=x} 2tdt = x^2 = 0(1) + 0(x) + 1(x^2) + 0(x^3) = 0\overrightarrow{\gamma_1} + 0\overrightarrow{\gamma_2} + \frac{1}{2}\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$

$$I_{[0,x]}(\overrightarrow{\delta_3}) = I_{[0,x]}(3x^2) = \int_{t=0}^{t=x} 3t^2dt = x^3 = 0(1) + 0(x) + 0(x^2) + 1(x^3) = 0\overrightarrow{\gamma_1} + 0\overrightarrow{\gamma_2} + 0\overrightarrow{\gamma_3} + \frac{1}{3}\overrightarrow{\gamma_4}$$

As in the previous problem, we find the representations of these output vectors in the basis γ.

The results are $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\delta_1})\right) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\delta_2})\right) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ and $Rep_\gamma\left(I_{[0,x]}(\overrightarrow{\delta_3})\right) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

Finally, we use those three columns to build a matrix that has three columns: $Rep_{\delta,\gamma}(I_{[0,x]}) = \begin{pmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

[3] Let $eval_5$ denote the "evaluation at 5 map, restricted to the 3rd degree polynomials". That is, $eval_5$ is the map $eval_5\colon \mathcal{P}_3 \to \mathbb{R}$ defined by $eval_5(f) = f(5)$. Represent the map $eval_5$ with respect to the bases γ and α.

Solution: We start by computing the outputs that result when the basis vectors of basis γ are used as input.

$$eval_5(\overrightarrow{\gamma_1}) = eval_5(1) = 1 = 1(1) = 1\overrightarrow{\alpha_1}$$
$$eval_5(\overrightarrow{\gamma_2}) = eval_5(x) = 5 = 5(1) = 5\overrightarrow{\alpha_1}$$
$$eval_5(\overrightarrow{\gamma_3}) = eval_5(x^2) = 5^2 = 25 = 25(1) = 25\overrightarrow{\alpha_1}$$
$$eval_5(\overrightarrow{\gamma_4}) = eval_5(x^3) = 5^3 = 125 = 125(1) = 125\overrightarrow{\alpha_1}$$

Now we find the representations of these output vectors in the basis α. The results are $Rep_\alpha\left(eval_5(\overrightarrow{\gamma_1})\right) = (1)$ and $Rep_\alpha\left(eval_5(\overrightarrow{\gamma_2})\right) = (5)$ and $Rep_\alpha\left(eval_5(\overrightarrow{\gamma_3})\right) = (25)$ and $Rep_\alpha\left(eval_5(\overrightarrow{\gamma_4})\right) = (125)$

Finally, we use those four column vectors to build a four-column matrix. $Rep_{\gamma,\alpha}(eval_5) = \begin{pmatrix} 1 & 5 & 25 & 125 \end{pmatrix}$

[4] Let $h: \mathcal{P}_2 \to \mathcal{P}_3$ be a linear map such that $h(1) = 2 + x^2$ and $h(x) = 1 - 3x$ and $h(x^2) = 5 - x + 7x^2$, and let $\vec{v} = 2 - 2x + 5x^2 \in \mathcal{P}_2$. The goal is to find $h(\vec{v})$ using matrix operations.

(a) Find the matrix $Rep_{\beta,\gamma}(h)$.

Solution: We start by computing the outputs that result when the basis vectors of basis β are used as input.

$$h(\overrightarrow{\beta_1}) = h(1) = 2 + x^2 = 2(1) + 0(x) + 1(x^2) + 0(x^3) = 2\overrightarrow{\gamma_1} + 0\overrightarrow{\gamma_2} + 1\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$
$$h(\overrightarrow{\beta_2}) = h(x) = 1 - 3x = 1(1) - 3(x) + 0(x^2) + 0(x^3) = 1\overrightarrow{\gamma_1} - 3\overrightarrow{\gamma_2} + 0\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$
$$h(\overrightarrow{\beta_3}) = h(x^2) = 5 - x + 7x^2 = 5(1) - 1(x) + 7(x^2) + 0(x^3) = 5\overrightarrow{\gamma_1} - 1\overrightarrow{\gamma_2} + 7\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4}$$

As in the previous problems, we find the representations of these output vectors in the basis γ.

The results are $Rep_\gamma\left(h(\overrightarrow{\beta_1})\right) = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $Rep_\gamma\left(h(\overrightarrow{\beta_2})\right) = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix}$ and $Rep_\gamma\left(h(\overrightarrow{\beta_3})\right) = \begin{pmatrix} 5 \\ -1 \\ 7 \\ 0 \end{pmatrix}$.

Finally, we use those three column vectors to build a three-column matrix. $Rep_{\beta,\gamma}(h) = \begin{pmatrix} 2 & 1 & 5 \\ 0 & -3 & -1 \\ 1 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Find the column vector $Rep_\beta(\vec{v})$.

Solution: We must first express vector $\vec{v}$ as a linear combination of the basis vectors in basis β.

$$\vec{v} = 2 - 2x + 5x^2 = 2(1) - 2(x) + 5(x^2) = 2\overrightarrow{\beta_1} - 2\overrightarrow{\beta_1} + 5\overrightarrow{\beta_3}$$

Now we can find the representation

$$Rep_\beta(\vec{v}) = Rep_\beta\left(2\overrightarrow{\beta_1} - 2\overrightarrow{\beta_1} + 5\overrightarrow{\beta_3}\right) = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$$

(c) Using the matrix-vector product, find $Rep_\gamma(h(\vec{v})) = Rep_{\beta,\gamma}(h) \cdot Rep_\beta(\vec{v})$.

Solution:

$$Rep_\gamma(h(\vec{v})) = Rep_{\beta,\gamma}(h) \cdot Rep_\beta(\vec{v}) = \begin{pmatrix} 2 & 1 & 5 \\ 0 & -3 & -1 \\ 1 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2(2) + 1(-2) + 5(5) \\ 0(2) - 3(-2) - 1(5) \\ 1(2) + 0(-2) + 7(5) \\ 0(2) + 0(-2) + 0(5) \end{pmatrix} = \begin{pmatrix} 27 \\ 1 \\ 37 \\ 0 \end{pmatrix}$$

(d) Using your result from (c), find $h(\vec{v})$. **Solution:** We use the fact that $h(\vec{v}) = Rep_\gamma{}^{-1}\left(Rep_\gamma(h(\vec{v}))\right)$. So

$$h(\vec{v}) = Rep_\gamma{}^{-1}\begin{pmatrix} 27 \\ 1 \\ 37 \\ 0 \end{pmatrix} = 27\overrightarrow{\gamma_1} + 1\overrightarrow{\gamma_2} + 37\overrightarrow{\gamma_3} + 0\overrightarrow{\gamma_4} = 27(1) + 1(x) + 37(x^2) + 0(x^3) = 27 + x + 37x^2$$

[5] Suppose that $Rep_{\beta,\delta}(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Observe that this is the 3×3 identity matrix.

(a) What is the domain of map f? What is the codomain of f?

Solution:

- The symbol $Rep_{\beta,\delta}(f)$ tells us that β is a basis for the domain space. At the start of this assignment, we are told that β is the standard bases for $\mathcal{P}_2$. Conclude that the domain of f must be $\mathcal{P}_2$.
- The symbol $Rep_{\beta,\delta}(f)$ tells us that δ is a basis for the codomain space. At the start of this assignment, we are told that δ is a non-standard bases for $\mathcal{P}_2$. Conclude that the codomain of f must be $\mathcal{P}_2$.

(b) Find $f(2 + 5x - x^2)$. The result should be a *function*.

Solution:

We start by representing the input vector $\vec{v} = 2 + 5x - x^2$ in the basis β.
We must first express vector $\vec{v}$ as a linear combination of the basis vectors in basis β.

$$\vec{v} = 2 + 5x - x^2 = 2(1) + 5(x) - (x^2) = 2\overrightarrow{\beta_1} + 5\overrightarrow{\beta_1} - \overrightarrow{\beta_3}$$

Now we can find the representation $Rep_\beta(\vec{v}) = Rep_\beta\left(2\overrightarrow{\beta_1} + 5\overrightarrow{\beta_1} - \overrightarrow{\beta_3}\right) = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

Now we will use the matrix-vector product to find $Rep_\delta\left(f(\vec{v})\right)$.

$$Rep_\delta\left(f(\vec{v})\right) = Rep_{\beta,\delta}(f) \cdot Rep_\beta(\vec{v}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Finally, we use the fact that $f(\vec{v}) = {Rep_\delta}^{-1}\left(Rep_\delta\left(f(\vec{v})\right)\right)$. So

$$f(\vec{v}) = {Rep_\delta}^{-1}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = 2\overrightarrow{\delta_1} + 5\overrightarrow{\delta_2} - 1\overrightarrow{\delta_3} = 2(1) + 5(2x) - 1(3x^2) = 2 + 10x - 3x^2$$

(Observe that f is not the identity map, because $f(2 + 5x - x^2) \neq 2 + 5x - x^2$!)

Now let $id: \mathcal{P}_2 \to \mathcal{P}_2$ be the identity map. That is, $id(a + bx + cx^2) = a + bx + cx^2$.

(c) Find $Rep_{\beta,\beta}(id)$. **Solution:** We start by computing the outputs when the vectors of basis β are used as input.

$$id\left(\overrightarrow{\beta_1}\right) = \overrightarrow{\beta_1} = 1\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}$$
$$id\left(\overrightarrow{\beta_2}\right) = \overrightarrow{\beta_2} = 0\overrightarrow{\beta_1} + 1\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3}$$
$$id\left(\overrightarrow{\beta_3}\right) = \overrightarrow{\beta_3} = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 1\overrightarrow{\beta_3}$$

As in the previous problems, we find the representations of these output vectors in the basis β.
The results are $Rep_\beta\left(id\left(\overrightarrow{\beta_1}\right)\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Rep_\beta\left(id\left(\overrightarrow{\beta_2}\right)\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $Rep_\beta\left(id\left(\overrightarrow{\beta_3}\right)\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Finally, we use those three column vectors to build a three-column matrix: $Rep_{\beta,\beta}(id) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Observe that the result *is* the 3×3 identity matrix.

(d) Find $Rep_{\beta,\delta}(id)$. (Observe that the result is not the 3×3 identity matrix!)
Solution: We start by computing the outputs when the vectors of basis β are used as input.

$$id\left(\overrightarrow{\beta_1}\right) = id(1) = 1 = 1(1) + 0(2x) + 0(3x^2) = 1\overrightarrow{\delta_1} + 0\overrightarrow{\delta_2} + 0\overrightarrow{\delta_3}$$
$$id\left(\overrightarrow{\beta_2}\right) = id(x) = x = 0(1) + \frac{1}{2}(2x) + 0(3x^2) = 0\overrightarrow{\delta_1} + \frac{1}{2}\overrightarrow{\delta_2} + 0\overrightarrow{\delta_3}$$
$$id\left(\overrightarrow{\beta_3}\right) = id(x^2) = x^2 = 0(1) + 0(2x) + \frac{1}{3}(3x^2) = 0\overrightarrow{\delta_1} + 0\overrightarrow{\delta_2} + \frac{1}{3}\overrightarrow{\delta_3}$$

As in the previous problems, we find the representations of these output vectors in the basis δ.
The results are $Rep_\delta\left(id\left(\overrightarrow{\beta_1}\right)\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Rep_\delta\left(id\left(\overrightarrow{\beta_2}\right)\right) = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$ and $Rep_\delta\left(id\left(\overrightarrow{\beta_3}\right)\right) = \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}$.

Finally, we use those three column vectors to build a three-column matrix: $Rep_{\beta,\delta}(id) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

Observe that the result *is not* the 3×3 identity matrix!