[1] Prove that the map $trace: \mathcal{M}_{m\times m} \to \mathbb{R}$ defined by $trace(M) = \sum_{i=1,\cdots,m} M_{i,i}$ is a linear map.
Solution: I will show that the trace map preserves all linear combinations.
Suppose that $A, B \in \mathcal{M}_{n\times n}$ and $c, d \in \mathbb{R}$. Then

$$\begin{aligned}
trace(cA+dB) &= \sum_{i=1,\cdots,m} (cA+dB)_{i,i} && \text{by definition of trace}\\
&= \sum_{i=1,\cdots,m} (cA)_{i,i} + (B)_{i,i} && \text{by definition of matrix addition}\\
&= \sum_{i=1,\cdots,m} c(A)_{i,i} + d(B)_{i,i} && \text{by definition of scalar multiplication}\\
&= \sum_{i=1,\cdots,m} c(A)_{i,i} + \sum_{i=1,\cdots,n} d(B)_{i,i} && \text{associative law for real number addition}\\
&= c\left(\sum_{i=1,\cdots,m} (A)_{i,i}\right) + d\left(\sum_{i=1,\cdots,m} (B)_{i,i}\right) && \text{distributive law for real numbers}\\
&= c\cdot trace(A) + d\cdot trace(B) && \text{by definition of trace}
\end{aligned}$$

[2] Suppose that $g: V \to W$ is linear and that $h: W \to Y$ is linear. Prove that $h \circ g$ is linear. Justify each step.
Solution: I will show that the composition preserves all linear combinations.
Suppose that $\overrightarrow{v_1}, \overrightarrow{v_2} \in V$ and $c_1, c_2 \in \mathbb{R}$. Then

$$\begin{aligned}
(h\circ g)(c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}) &= g\big(f(c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2})\big) && \text{by definition of composition}\\
&= g\big(c_1 f(\overrightarrow{v_1}) + c_2 f(\overrightarrow{v_2})\big) && \text{because } f \text{ is linear}\\
&= c_1 g\big(f(\overrightarrow{v_1})\big) + c_2 g\big(f(\overrightarrow{v_2})\big) && \text{because } f \text{ is linear}\\
&= c_1 (g\circ f)(\overrightarrow{v_1}) + c_2 (g\circ f)(\overrightarrow{v_2}) && \text{by definition of composition}
\end{aligned}$$

[3] Def.2.3 (p.226) gives a formula for the *i,j* entry of a matrix product: $p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$
Rewrite the formula using summation notation. Be sure to indicate what values the indices i and j can have.
(You will have to study the full presentation of the formula given in the definition to figure this out.)
Solution: The summation formula is below. The formula is valid for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j} = \sum_{k=1,\cdots,r} g_{i,k}h_{k,j}$$

Compute, or state "not defined". (Show details of all computations.)
Solution:

[4] $\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3(2)+1(1) & 3(0)+1(0) \\ -4(2)+2(1) & -4(0)+2(0) \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -6 & 0 \end{pmatrix}$

[5] $\begin{pmatrix} 1 & 3 & 6 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -2 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1(-1)+3(-2)+6(1) & 1(-3)+3(5)+6(-2) \\ 2(-1)+3(-2)+4(1) & 2(-3)+3(5)+4(-2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

[6] $\begin{pmatrix} 1 & 2 \\ 3 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -2 & 5 \\ 1 & -2 \end{pmatrix}$ because the product of a 2×3 matrix and a 2×3 matrix is not defined.

[7] $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3(1)+(-1)(2) & 3(1)+(-1)(3) \\ (-2)(1)+1(2) & (-2)(1)+1(3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[8] Define the map $D: \mathcal{P}_4 \to \mathcal{P}_4$ by $D(f) = \frac{df}{dx}$. Let $\beta = \langle 1, x, x^2, x^3, x^4 \rangle$ be the standard basis for $\mathcal{P}_4$.

(a) Find the matrix $M = Rep_{\beta,\beta}(D)$.
Solution: We start by computing the outputs when the vectors of basis γ are used as input.

$$D(\overrightarrow{\beta_1}) = D(1) = \frac{d}{dx}(1) = 0 = 0(1) + 0(x) + 0(x^2) + 0(x^3) + 0(x^4) = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3} + 0\overrightarrow{\beta_4} + 0\overrightarrow{\beta_5}$$
$$D(\overrightarrow{\beta_2}) = D(x) = \frac{d}{dx}(x) = 1 = 1(1) + 0(x) + 0(x^2) + 0(x^3) + 0(x^4) = 1\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3} + 0\overrightarrow{\beta_4} + 0\overrightarrow{\beta_5}$$
$$D(\overrightarrow{\beta_3}) = D(x^2) = \frac{d}{dx}(x^2) = 2x = 0(1) + 2(x) + 0(x^2) + 0(x^3) + 0(x^4) = 0\overrightarrow{\beta_1} + 2\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3} + 0\overrightarrow{\beta_4} + 0\overrightarrow{\beta_5}$$
$$D(\overrightarrow{\beta_4}) = D(x^3) = \frac{d}{dx}(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2) + 0(x^3) + 0(x^4) = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 3\overrightarrow{\beta_3} + 0\overrightarrow{\beta_4} + 0\overrightarrow{\beta_5}$$
$$D(\overrightarrow{\beta_5}) = D(x^4) = \frac{d}{dx}(x^4) = 4x^3 = 0(1) + 0(x) + 0(x^2) + 4(x^3) + 0(x^4) = 0\overrightarrow{\beta_1} + 0\overrightarrow{\beta_2} + 0\overrightarrow{\beta_3} + 4\overrightarrow{\beta_4} + 0\overrightarrow{\beta_5}$$

Now we find the representations of these output vectors in the basis β. The results are:

$$Rep_\beta\left(D(\overrightarrow{\beta_1})\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, Rep_\beta\left(D(\overrightarrow{\beta_2})\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, Rep_\beta\left(D(\overrightarrow{\beta_3})\right) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$Rep_\beta\left(D(\overrightarrow{\beta_4})\right) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, Rep_\beta\left(D(\overrightarrow{\beta_5})\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

Finally, we use those five column vectors to build a matrix. $M = Rep_{\beta,\beta}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(b) Find the matrix $M^2 = M \cdot M$.

Solution: I will use the layout trick:

$$\begin{array}{cc} & (M) \\ (M) & (M \cdot M) \\ & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

So $M^2 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(c) What does M^2 represent?
Solution: You can probably guess that M^2 represents D^2, which would be the second derivative operator. But it is nice to see how that works out using official notation.

$$M^2 = M \cdot M = Rep_{\beta,\beta}(D) \cdot Rep_{\beta,\beta}(D) = Rep_{\beta,\beta}(D \circ D) = Rep_{\beta,\beta}(D^2)$$

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[9] Suppose that $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$. Using summation notation, show that $(AB)^T = B^T A^T$. Be sure to use correct upper and lower limits for all indices.

Solution: The equation expresses an equality of matrices. We prove that two matrices are equal by showing that every one of their entries is equal.

We should first confirm that the two matrices will be the same shape. If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, then

- The left matrix: Then the product AB will be $m \times p$, so $(AB)^T$ will be $p \times m$.
- The right matrix: B^T will be $p \times n$ and A^T will be $n \times m$, so $B^T A^T$ will be $p \times m$.

So the left and right matrices are the same size. Good.

Now let's show that their entries are equal. For any $i = 1 \ldots p$ and $j = 1 \ldots m$, the i, j^{th} entry will be

$$\begin{aligned}
((AB)^T)_{i,j} &= (AB)_{j,i} && \text{by definition of transpose} \\
&= \sum_{k=1,\cdots,n} A_{j,k} B_{k,i} && \text{by definition of matrix multiplication} \\
&= \sum_{k=1,\cdots,n} B_{k,i} A_{j,k} && \text{by commutativity of real number multiplication} \\
&= \sum_{k=1,\cdots,n} (B^T)_{i,k} (A^T)_{k,j} && \text{by definition of transpose} \\
&= (B^T A^T)_{i,j} && \text{by definition of matrix multiplication}
\end{aligned}$$

Since all of the entries are equal, we condlude that $(AB)^T = B^T A^T$.

[10] Suppose that $A \in \mathcal{M}_{k\times p}$ and $B \in \mathcal{M}_{p\times k}$. Using summation notation, show that $trace(AB) = trace(BA)$. Be sure to use correct upper and lower limits for all indices.

Solution: First, it is important to note the shapes of the products. Suppose that $A \in \mathcal{M}_{k\times p}$ and $B \in \mathcal{M}_{p\times k}$. The product AB will be $k \times k$, while the product BA will be $p \times p$. So products AB and BA are not the same shape.

$$\begin{aligned}
trace(AB) &= \sum_{i=1,\cdots,k} (AB)_{i,i} && \text{by definition of trace for } k \times k \text{ matrix} \\
&= \sum_{i=1,\cdots,k} \sum_{j=1,\cdots,p} A_{i,j} B_{j,i} && \text{by definition of matrix multiplication} \\
&= \sum_{i=1,\cdots,k} \sum_{j=1,\cdots,p} B_{j,i} A_{i,j} && \text{by commutative property of real number addition} \\
&= \sum_{j=1,\cdots,p} \sum_{i=1,\cdots,k} B_{j,i} A_{i,j} && \text{by associative and comm. properties of real number addition.} \\
&= \sum_{j=1,\cdots,p} (BA)_{j,j} && \text{by definition of matrix multiplication} \\
&= trace(BA) && \text{by definition of trace for } p \times p \text{ matrix}
\end{aligned}$$

Notice that it turned out to be okay that the products AB and BA are not the same shape.