Section 1: Image

Suppose that f is a function $f: A \to B$
If $x \in A$, then the symbol $f(x)$ denotes the output that results when x is used as input.
Notice that $f(x) \in B$.
Another name for the output $f(x)$ is "the image of x under the map f".

So the input x is an element of the domain A, while the image $f(x)$ is an element of the codomain B.

Examples
Example #1: For $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, we have the following:

- The image of 2 under the map f is 4, because $f(2) = 2^2 = 4$.
- The image of -2 under the map f is 4, because $f(-2) = (-2)^2 = 4$.

Example #2: For $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$, we have the following:

- The image of 2 under the map g is 8, because $g(2) = 2^3 = 8$.
- The image of -2 under the map g is -8, because $\mathrm{g}(-2) = (-2)^3 = -8$.

Section 2: Preimage

Suppose that f is a function $f: A \to B$
If $y \in B$, then the symbol $f^{-1}(y)$ denotes the set of all inputs that will yield y as an output.
Notice that $f^{-1}(y) \subset A$.
Another name for the set $f^{-1}(y) \subset A$ is "the preimage of y under the map f".

So the output y is an element of the codomain B, while the preimage $f^{-1}(y)$ is a subset of the domain A.

Examples:
Example #1: For $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, we have the following:

- The preimage of 4 under the map f is the set $\{-2,2\}$, because $f(2) = 4$ and $f(-2) = 4$
- The preimage of 0 under the map f is the set $\{0\}$, because only $f(0) = 0$.
- The preimage of -5 under the map f is the the empty set ϕ, because there is no x such that $x^2 = -5$

Using the notation for the preimage, we would write

- $f^{-1}(4) = \{-2,2\}$.
- $f^{-1}(0) = \{0\}$.
- $f^{-1}(-5) = \phi$.

Again notice that in each case, the preimage is a subset of the domain.

Example #2: For $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$, we have the following:

- The preimage of 8 under the map g is the set $\{2\}$, because only $g(2) = 8$.
- The preimage of 0 under the map g is the set $\{0\}$, because only $g(0) = 0$.

- The preimage of -5 under map g is the the set $\{-(5)^{1/3}\}$, because $g\left(-(5)^{1/3}\right) = \left(-(5)^{1/3}\right)^3 = -5$

Using the notation for the preimage, we would write

- $g^{-1}(8) = \{2\}$.
- $g^{-1}(0) = \{0\}$.
- $g^{-1}(-5) = \{-(5)^{1/3}\}$.

Again notice that in each case, the preimage is a subset of the domain.

Section 3: Inverse Functions

First, a definition

Definition of inverse functions:
words: f and g are inverse functions.
meaning: f and g are functions satisfy the following four conditions

- $f: A \to B$ for some domain A and some codomain B.
- $g: B \to A$. That is, the domain and codomain of g are the reverse of what they are for f.
- For all $x \in A$, the equation $g(f(x)) = x$ is true.
- For all $y \in B$, the equation $f(g(y)) = y$ is true.

The last two conditions are called the *inverse relations*.

Additional terminology: If : f and g are inverse functions, we also say that g is an inverse function for f, and we also say that f is an inverse function for g.

Examples

Example #1: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = x^3$, and let $g: \mathbb{R} \to \mathbb{R}$ be $g(x) = x^{1/3}$.
Observe that $g(f(x)) = (x^3)^{1/3} = x$.
Also observe that $f(g(y)) = \left((y)^{1/3}\right)^3 = y$.
So the inverse relations are both true. Conclude that f and g are inverse functions.

Example #2: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f does not have an inverse function. To see why, consider what happens when we try to come up with one. Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$.

- Observe that $g(f(2)) = \sqrt{2^2} = 2$, which is fine, but $g(f(-2)) = \sqrt{(-2)^2} = 2$. So the equation $g(f(x)) = x$ is not always true.
- Also observe that $f(g(5)) = \left(\sqrt{5}\right)^2 = 5$ which is fine, but $f(g(-5)) = \left(\sqrt{-5}\right)^2$ which does not even exist. So the equation $f(g(y)) = y$ is not always true.

Facts about inverse functions:

- A function $f: A \to B$ has an inverse function $g: B \to A$ if and only if f is one-to-one and onto.
- The inverse function g will also be one-to-one and onto.
- If some functions $f: A \to B$ and $g: B \to A$ satisfy both inverse relations:
  - For all $x \in A$, the equation $g(f(x)) = x$ is true.

  - For all $y \in B$, the equation $f(g(y)) = y$ is true.

  then it can be proven that f and g are both one-to-one and onto, so they qualify to be called inverse functions.
- Inverse functions are unique: A function can have only one inverse function.

Additional notation: If function $f: A \to B$ has an inverse function, we use the symbol f^{-1} to denote the unique inverse function.

Example:
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Then f has an inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = y^{1/3}$.

Section 4: Using inverse notation:

Observe that inverse notation and preimage notation look the same. This is confusing. I will discuss some examples:

Example #1
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. What does the symbol $f^{-1}(8)$ mean? There are two possibilities.

- Remember that in Section 2 above, the symbol $f^{-1}(8)$ meant the preimage of 8 under the map f. The preimage is always a set, a subset of the domain. We wrote $f^{-1}(8) = \{2\}$. Notice that this is a set.
- But in Section 3, the symbol f^{-1} was used to denote the inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}$ defined by the formula $f^{-1}(y) = y^{1/3}$. In that usage, the symbol $f^{-1}(8)$ would denote the output that results when we feed the number y into the function f^{-1}. That is $f^{-1}(8) = (8)^{1/3} = 2$. Notice that this is a number, not a set.

Which interpretation of the meaning of the symbol $f^{-1}(8)$ is correct? Generally, if a function f has an inverse function, then we interpret the symbol $f^{-1}(8)$ to mean the number that results when an input of 8 is fed into the inverse function.. That is, we interpret the symbol $f^{-1}(8)$ to mean a single element of the domain.

Example #2
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. What does the symbol $f^{-1}(4)$ mean? There is only one possibility.

- As in Section 2 above, the symbol $f^{-1}(4)$ means the preimage of 4 under the map f. The preimage is always a set, a subset of the domain. In this case, it is $f^{-1}(4) = \{2, -2\}$.
- Because $f(x) = x^2$ has no inverse function, the symbol $f^{-1}(4)$ cannot be interpreted in terms of an inverse function. It can only be interpreted as a symbol for a preimage.

More Examples:

[1] Define map $f: \mathcal{P}_2 \to \mathbb{R}^3$ by $f(a+bx+cx^2) = \begin{pmatrix} b \\ a-b \\ b+c \end{pmatrix}$.

The book would write $a+bx+cx^2 \xrightarrow{f} \begin{pmatrix} b \\ a-b \\ b+c \end{pmatrix}$

Find the image of each of these elements of the domain: (a) $\vec{v_1} = 2-3x+4x^2$ (b) $\vec{v_2} = x+x^2$

Solution: (a) The image of $\vec{v_1}$ is the vector $f(\vec{v_1}) = f(2-3x+4x^2) = \begin{pmatrix} -3 \\ 2-(-3) \\ (-3)+4 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$.

(b) The image of $\vec{v_1}$ is the vector $f(\vec{v_2}) = f(x+x^2) = \begin{pmatrix} 1 \\ 0-1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

[2] Consider the isomorphim $Rep_\beta: \mathcal{P}_2 \to \mathbb{R}^3$, where β is the basis $\beta = \langle \vec{w_1}, \vec{w_2}, \vec{w_3} \rangle = \langle 1, 1+x, 1+x+x^2 \rangle$ for $\mathcal{P}_2$. Find the image of each of these elements of the domain:
(a) $\vec{v_1} = 2-3x+4x^2$ (b) $\vec{v_2} = x+x^2$.

Solution:

First note that the representation map $Rep_\beta: \mathcal{P}_2 \to \mathbb{R}^3$ works in the following way:

$$Rep_\beta(c_1 \cdot \vec{w_1} + c_2 \cdot \vec{w_2} + c_3 \cdot \vec{w_3}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

In order to determine the output, the input vector must be expressed as a linear combination of the basis vectors $\vec{w_1}, \vec{w_2}, \vec{w_3}$ from basis β. So we start by expressing $\vec{v_1}$ and $\vec{v_2}$ as linear combinations of $\vec{w_1}, \vec{w_2}, \vec{w_3}$.

(a) $\vec{v_1} = 2-3x+4x^2 = (5)(1) + (-7)(1+x) + 4(1+x+x^2) = 5\vec{w_1} - 7\vec{w_2} + 4\vec{w_3}$

(b) $\vec{v_2} = x+x^2 = (-1)(1) + (0)(1+x) + 1(1+x+x^2) = (-1)\vec{w_1} + (0)\vec{w_2} + (1)\vec{w_3}$

Now that we know those linear combinations, we can compute the representations.

(a) $Rep_\beta(\vec{v_1}) = Rep_\beta(5\vec{w_1} - 7\vec{w_2} + 4\vec{w_3}) = \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}$

(b) $Rep_\beta(\vec{v_2}) = Rep_\beta\big((-1)\vec{w_1} + (0)\vec{w_2} + (1)\vec{w_3}\big) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

[3] Every isomorphism has an inverse. For the isomorphism in problem [2], find ${Rep_\beta}^{-1}\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

Solution:

First note that the inverse map ${Rep_\beta}^{-1}: \mathbb{R}^3 \to \mathcal{P}_2$ works as the reverse of the map Rep_β as follows:

$${Rep_\beta}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \cdot \vec{w_1} + c_2 \cdot \vec{w_2} + c_3 \cdot \vec{w_3}$$

Therefore,

$${Rep_\beta}^{-1}\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 2\vec{w_1} + (-1)\vec{w_2} + 5\vec{w_3} = 2(1) - (1+x) + 5(1+x+x^2) = 6+4x+5x^2.$$