Solutions to Induction Problems about Inequalities

2017-2018 Spring Semester MATH 3050 Section 101 (Barsamian) Homework 6

Prove Statement A: $\forall n \in Z, n \geq 4(n^3 < 3^n)$.

n	n^3	3^n	$n!$
0	0	1	1
1	1	3	1
2	8	9	2
3	27	27	6
4	64	81	24
5	125	243	120
6	216	729	720
7	343	2187	5040
8	512	6561	40320
9	729	19683	362880
10	1000	59049	3.6288×10^6
11	1331	177147	3.99168×10^7
12	1728	531441	4.790016×10^8

Preliminary work:

Starting value: $a = 4$

Predicate: $P(n)$ is the inequality $n^3 < 3^n$.

Predicate with k inside: $P(k)$ is the inequality $k^3 < 3^k$.

Predicate with $k+1$ inside: $P(k+1)$ is $(k+1)^3 < 3^{(k+1)}$.

Basis Step: $P(4)$ is the inequality $4^3 < 3^4$. That is, $64 < 81$. True.

Induction Step

Suppose that $k \geq 4$ and that $P(k)$ is true.

That is, $k \geq 4$ and $k^3 < 3^k$ is true. (This is the Inductive Hypothesis)

Start by building the left side of the inequality $P(k+1)$ and show that it is less than the right side.

$$\begin{aligned}
left\ side &= (k+1)^3 \\
&= k^3 + 3k^2 + 3k + 1 \ \ (expanded) \\
&< k^3 + 3k^2 + 3k + 3k \ (because\ 1 < 3k\ since\ 4 \leq k) \\
&= k^3 + 3k^2 + 3(2k) \ (arithmetic) \\
&< k^3 + 3k^2 + 3(k^2) \ (because\ 2 < k\ since\ 4 \leq k) \\
&< k^3 + k^3 + k^3 \ \ (because\ 3 < k\ since\ 4 \leq k) \\
&= 3 \cdot k^3 \ \ (factored) \\
&< 3 \cdot 3^k \ \ (because\ k^3 < 3^k\ by\ the\ Inductive\ Hypothesis) \\
&= 3^{(k+1)} \ \ (rule\ of\ exponents) \\
&= right\ side
\end{aligned}$$

We have shown that $P(k+1)$ is true.

Conclusion: By the Principle of Induction, $\forall n \in Z, n \geq 4(n^3 < 3^n)$

Prove Statement B: $\forall n \in Z, n \geq 6(n^3 < n!)$.

Preliminary work:

Starting value: $a = 6$

Predicate: $P(n)$ is the inequality $n^3 < n!$.

Predicate with k inside: $P(k)$ is the inequality $k^3 < k!$.

Predicate with $k+1$ inside: $P(k+1)$ is the inequality $(k+1)^3 < (k+1)!$

Basis Step: $P(6)$ is the inequality $6^3 < (6+1)!$. That is, $216 < 720$. This is true.

Induction Step

Suppose that $k \geq 6$ and that $P(k)$ is true. That is, $k^3 < k!$ is true. (This is the Inductive Hypothesis)

Start by building the left side of the inequality $P(k+1)$ and show that it is less than the right side.

$$\begin{aligned} left\ side &= (k+1)^3 \\ &= k^3 + 3k^2 + 3k + 1 \ \ (expanded) \\ &< k^3 + 3k^2 + 3k + 3k \ (because\ 1 < 3k\ since\ 6 \leq k) \\ &= k^3 + 3k^2 + 3(2k)\ (arithmetic) \\ &< k^3 + 3k^2 + 3(k^2)\ (because\ 2 < k\ since\ 6 \leq k) \\ &< k^3 + k^3 + k^3 \ \ (because\ 3 < k\ since\ 6 \leq k) \\ &= 3 \cdot k^3 \ \ (factored) \\ &< 3 \cdot k! \ \ (because\ k^3 < k!\ by\ the\ Inductive\ Hypothesis) \\ &< (k+1) \cdot k! \ \ (because\ 3 < (k+1)\ since\ 6 \leq k) \\ &= (k+1)! \ \ (property\ of\ factorial) \\ &= right\ side \end{aligned}$$

We have shown that $P(k+1)$ is true.

Conclusion: By the Principle of Induction, $\forall n \in Z, n \geq 6(n^3 < n!)$

Prove Statement *C*: $\forall \boldsymbol{n} \in \boldsymbol{Z}, \boldsymbol{n} \geq \mathbf{7}(\mathbf{3}^{\boldsymbol{n}} < \boldsymbol{n}!)$.

Preliminary work:

Starting value: $a = 7$

Predicate: $P(n)$ is the inequality $3^n < n!$.

Predicate with k inside: $P(k)$ is the inequality $3^n < k!$.

Predicate with $k+1$ inside: $P(k+1)$ is the inequality $3^{(k+1)} < (k+1)!$

Basis Step: $P(7)$ is the inequality $7^3 < (7+1)!$. That is, $2187 < 5040$. This is true.

Induction Step

Suppose that $k \geq 7$ and that $P(k)$ is true. That is, $3^k < k!$ is true. (This is the Inductive Hypothesis)

Start by building the left side of the inequality $P(k+1)$ and show that it is less than the right side.

$$\begin{aligned} left\ side &= 3^{(k+1)} \\ &= 3 \cdot 3^k \ \ (rule\ of\ exponents) \\ &< 3 \cdot k! \ \ (because\ 3^k < k!\ by\ the\ Inductive\ Hypothesis) \\ &< (k+1) \cdot k! \ \ (because\ 3 < (k+1)\ since\ 7 \leq k) \\ &= (k+1)! \ \ (property\ of\ factorial) \\ &= right\ side \end{aligned}$$

We have shown that $P(k+1)$ is true.

Conclusion: By the Principle of Induction, $\forall n \in Z, n \geq 7(3^n < n!)$.