Do the symbols $5/7$ and $5|7$ mean the same thing? Explain.

Do the symbols $2/6$ and $2|6$ mean the same thing? Explain.

Frick and Frack are arguing. Frick says that $3/0$ is undefined. Frack says that $3|0$ is true. Who is right? Explain.

Sonny and Cher are arguing. Sonny says that $0/5$ is zero. Cher says that $0|5$ is false. Who is right? Explain.

Is 24 divisible by 3? Explain. Does 8 divide 40? Explain.

Does $6|42$? Does $42|6$? Does $6|6$? Does $6|(-24)$? Explain.

For what n does $1|n$? For what n does $n|1$? Explain.

For what n does $0|n$? For what n does $n|0$? Explain.

$5/7$ is a real number

$5 \mid 7$ is the statement

- 5 divides 7 (false)
- $\frac{7}{5}$ is an integer (false)

$2/6$ is a real number

$2 \mid 6$ is the statement

- 2 divides 6 (true statement)
- $\frac{6}{2}$ is an integer (true statement about $\frac{6}{2}$)

Frick: $\frac{3}{0}$ is undefined True

Frack: $3 \mid 0$ is true True

$\left(\frac{0}{3} \text{ is an integer} \right)$ is true true

$\left(0 = 3 \cdot k \text{ for some integer } k \right)$ is true true

let $k = 0$

$\frac{0}{5}$ is the real number 0 true

$0 \mid 5$ is false

$\frac{5}{0}$ is an integer is false True

$1 \mid n$ is true for any integer n

$1 \mid n$ means $\frac{n}{1}$ is an integer. True

$n \mid 1$ means $\frac{1}{n}$ is an integer true when $n = 1$ or $n = -1$

$0 \mid n$ means $\frac{n}{0}$ is an integer never true!

$n \mid 0$ means $\frac{0}{n}$ is an integer true for all $n \neq 0$ false for $n = 0$.

Question 1: In division, why not just declare that $\frac{0}{0} = 1$?

Question 2: Observe that $0 = 0 \cdot 1$. Since $0 = 0 \cdot integer$, one might think that this would imply that $0|0$ is true. But the definition says that we are not allowed to have 0 on the left side of the vertical bar. That is, $0|anything$ is never true. Not even $0|0$. Why not?

The answers to both questions are fundamental and have to do with the

Axioms of Real Numbers (From Appendix A1)

We assume that there are two binary operations defined on the set of real numbers, called **addition** and **multiplication**, such that if a and b are any two real numbers, the **sum** of a and b, denoted $\boldsymbol{a+b}$, and the **product** of a and b, denoted $\boldsymbol{a \cdot b}$ or $\boldsymbol{ab}$, are also real numbers. These operations satisfy properties F1–F6, which are called the **field axioms**.

F1. *Commutative Laws* For all real numbers a and b,

$$a+b=b+a \quad \text{and} \quad ab=ba.$$

F2. *Associative Laws* For all real numbers a, b, and c,

$$(a+b)+c=a+(b+c) \quad \text{and} \quad (ab)c=a(bc).$$

F3. *Distributive Laws* For all real numbers a, b, and c,

$$a(b+c)=ab+ac \quad \text{and} \quad (b+c)a=ba+ca.$$

F4. *Existence of Identity Elements* There exist two distinct real numbers, denoted 0 and 1, such that for every real number a,

$$0+a=a+0=a \quad \text{and} \quad 1\cdot a=a\cdot 1=a.$$

F5. *Existence of Additive Inverses* For every real number a, there is a real number, denoted $-a$ and called the **additive inverse** of a, such that

$$a+(-a)=(-a)+a=0.$$

F6. *Existence of Reciprocals* For every real number $a \neq 0$, there is a real number, denoted $1/a$ or a^{-1}, called the **reciprocal** of a, such that

$$a\cdot\left(\frac{1}{a}\right)=\left(\frac{1}{a}\right)\cdot a=1.$$

The Axioms of Real Numbers guarantee that every real number $a \neq 0$ has a reciprocal, denoted $1/a$ or $\frac{1}{a}$ or a^{-1}.

To say that

$$\frac{0}{0} = 1 \qquad \frac{0}{0} = 1$$

Really means that

$$0 \cdot \left(\frac{1}{0}\right) = 1 \qquad 0 \cdot \left(\frac{1}{0}\right) = 1$$

That is, it means that the number 0 has a reciprocal.

The axioms don't guarantee that the number $a = 0$ has a reciprocal, but they don't explicitly forbid it. Is it possible that 0 can have a reciprocal, even though it is not guaranteed by the axioms?

The answer lies in the *Theorems of Real Numbers* that immediately follow the Axioms.

Theorems of Real Numbers (Just the first six) (From Appendix A1)

(These theorems are statements that are not included in the list of Axioms because they don't have to be. They can be proven true as a consequence of the Axioms.)

All the usual algebraic properties of the real numbers that do not involve order can be derived from the field axioms. The most important are collected as theorems T1–T16 as follows. In all these theorems the symbols a, b, c, and d represent arbitrary real numbers.

T1. *Cancellation Law for Addition* If $a+b=a+c$, then $b=c$. (In particular, this shows that the number 0 of Axiom F4 is unique.)

T2. *Possibility of Subtraction* Given a and b, there is exactly one x such that $a+x=b$. This x is denoted by $b-a$. In particular, $0-a$ is the additive inverse of a, $-a$.

T3. $b-a=b+(-a)$.

T4. $-(-a)=a$.

T5. $a(b-c)=ab-ac$.

T6. $0\cdot a=a\cdot 0=0$.

T6 says $0 \cdot$ any number $= 0$

Notice that Theorem **T6** says explicitly

For any real number a, $0 \cdot a = 0$

So it is impossible for the number 0 to have a reciprocal. That is, it is impossible to have

$$\frac{0}{0} = 1$$

because that would violate **T6**.

And note that the problem with trying to declare

$$\frac{0}{0} = 1$$

is not just a problem because it would violate **T6**. It would cause problems in lots of other ways, as well.

$\frac{a}{b}$ means $a \cdot \left(\frac{1}{b}\right)$

For instance, consider this calculation

$$7 = 7 \cdot 1 = 7 \cdot \left(\frac{0}{0}\right) = 7 \cdot \left(0 \cdot \left(\frac{1}{0}\right)\right) = (7 \cdot 0) \cdot \left(\frac{1}{0}\right) = 0 \cdot \left(\frac{1}{0}\right) = \frac{0}{0} = 1$$

.

property of the number 1

because the guy who claims $\frac{0}{0} = 1$

definition of what the symbol $\frac{a}{b}$ means

by associativity

by Theorem T6

definition of what the symbol $\frac{a}{b}$ means

because that guy again

The Quotient Remainder Theorem Content from 4.5

Consider this collection of equations

an infinite collection of equations

all of these equations are true

$$\vdots$$

$$71 = 9 \cdot 10 + (-19)$$

$$71 = 9 \cdot 9 + (-10)$$

$$71 = 9 \cdot 8 + (-1)$$

$$71 = 9 \cdot 7 + 8$$ the special one. because $0 \le 8 < 9$

$$71 = 9 \cdot 6 + 17$$

$$71 = 9 \cdot 5 + 26$$

$$\vdots$$

$\vdots$

$$-71 = 9 \cdot (-10) + 19$$

$$-71 = 9 \cdot (-9) + 10$$

Special

$$-71 = 9 \cdot (-8) + 1 \qquad 0 \le 1 < 9$$

$$-71 = 9 \cdot (-7) + -8$$

$\vdots$

$\vdots$

$$72 = 9 \cdot (10) + (-18)$$

$$72 = 9 \cdot (9) + (-9)$$

$$72 = 9 \cdot (8) + 0 \qquad 0 \le 0 < 9$$

$$72 = 9 \cdot (7) + 9 \qquad \cancel{0 \le 9 < 9}$$

False

$\vdots$

The quotient remainder theorem (QRT)

For every integer n and every positive integer d there exists exactly one integer q and one integer r such that

$$n = d \cdot q + r \quad \text{and} \quad 0 \le r < d$$

$$\forall n \in \mathbb{Z} \left(\exists d \in \mathbb{Z}^+ \left(\exists q, r \in \mathbb{Z} \left((n = d \cdot q + r) \wedge (0 \le r < d) \right) \right) \right)$$

r is called the remainder

One more rule of inference

Proof by Division Into Cases (book page 76)

$$\boxed{\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \therefore r \end{array}}$$

Example will involve three cases

generalizes to more variables

Example of use of this, in conjunction with quotient remainder theorem

Let n be an integer.

What does quotient remainder theorem (QRT) say about $d=3$

There is exactly one $q, r \in \mathbb{Z}$ such that

$$n = 3q + r \quad \text{AND} \quad 0 \le r < 3$$

So either $n = 3q + 0$

or $n = 3q + 1$

$n = 3q + 2$

exactly one of these is true

Example Prove that the product of any 3 consecutive integers is divisible by 3.

Proof

(1) 3 consecutive integers can always be written

$$n, \quad n+1, \quad n+2$$

So their product is

$$n(n+1)(n+2)$$

(2) QRT with $d=3$ says

$$(n=3q) \text{ or } (n=3q+1) \text{ or } (n=3q+2)$$

Case 1

(3) ~~If~~ $n=3q$

Then $n(n+1)(n+2) = 3q(3q+1)(3q+2)$

which is divisible by 3

Case 2

(4) If $(n = 3q + 1)$

then $n(n+1)(n+2) = (3q+1)(3q+1+1)\underbrace{(3q+1+2)}_{\text{this is a multiple of 3}}$

this is divisible by 3

- Case 3

(5) If $n = 3q + 2$ then

$n(n+1)(n+2) = (3q+2)((3q+2)+1)((3q+2)+2)$

$= (3q+2)\underbrace{(3q+3)}_{\text{this is a multiple of 3}}(3q+4)$

this is divisible by 3

Conclusion

(6) Since $n(n+1)(n+2)$ is divisible by 3 in every case, we conclude that $n(n+1)(n+2)$ is divisible by 3.

End of proof