Section 2.2 Conditional Statements.

New Compound Statement Form: The *Conditional* (the *IF THEN* statement form)

Definition of the *Conditional*

symbols: $p \longrightarrow q$ IF p THEN q

spoken: if p then q

less commonly spoken: the conditional of q by p

usage: p, q are statement variables

meaning: $p \rightarrow q$ is a statement form whose truth values are defined by this truth table

truth of p	truth of q	truth of $p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

additional terminology: p is called the hypothesis
q is called the conclusion

[Example]

Which of these statements is true? Explain.

A: IF $\underbrace{\text{4 is an even integer}}_{T}$, THEN $\underbrace{\text{Thanksgiving is in November}}_{T}$. T → T T

B: IF $\underbrace{\text{4 is an even integer}}_{T}$, THEN $\underbrace{\text{the world is flat}}_{F}$. T → F F

C: IF $\underbrace{\text{the moon is made of green cheese}}_{F}$, THEN $\underbrace{\text{Thanksgiving is in November}}_{T}$. F → T T

D: IF $\underbrace{\text{the moon is made of green cheese}}_{F}$, THEN $\underbrace{\text{the world is flat}}_{F}$. F → F T

Remark: There is a special term for the situation where a conditional statement that is true by virtue of the fact that its *hypothesis* is false. The statement is said to be ***vacuously true***.

End of [Example]

Order of Operations

We have encountered four logical symbols so far: $\wedge, \vee, \rightarrow, \sim$.

(We won't be using the exclusive or symbol $\oplus$.)

What is the order of operations?

$(\ \ ), \sim, \wedge, \vee, \rightarrow$

That is,

Parentheses, NOT, AND, OR, Conditional.

[Example] Construct a truth table for the statement form $p \wedge \sim q \rightarrow r$

3 statement Variables, so we will need
$2 \cdot 2 \cdot 2 = 8$ rows

$(p \wedge (\sim q)) \rightarrow r$

p	q	r	$\sim q$	$p \wedge \sim q$	$(p \wedge (\sim q)) \rightarrow r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

End of [Example]

Express the Conditional Statement Form in an Alternate form

$(\sim p) \vee q$

[Example] Use a truth table to prove that $p \to q$ is logically equivalent to $\sim p \vee q$

Solution: need $2 \cdot 2 = 4$ rows because 2 statement variables p, q

p	q	~p	p → q	~p ∨ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

T ∨ F
F ∨ F
T ∨ T
F ∨ T

End of [Example] These two columns match, so the statement forms are logically equivalent.

Goal: Find the Negation of a conditional statement

Given that $p \to q \equiv \sim p \vee q$, use DeMorgan to find $\sim(p \to q)$.

$$\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv (\sim(\sim p)) \wedge \sim q \equiv p \wedge \sim q$$

The Negation of the Conditional Statement Form

For the conditional statement form ***S***: $p \to q$ IF p THEN q

The ***negation of S*** is the statement form: $p \wedge \sim q$ p AND NOT q

That is,

$$\sim(p \to q) \equiv p \wedge \sim q$$

Remark: The ***negation of a conditional statement form*** is an ***AND*** **statement form**

[Example]

Negate statement S: If $\overbrace{\text{today is Saturday}}^{p}$, then $\overbrace{\text{I go to Kroger today}}^{q}$.

$p \to q$

$$\sim(p \to q) \equiv p \wedge \sim q$$

Today is saturday and I do not go to Kroger today.

Negate statement S: If $\overbrace{\text{Bob is green}}^{p}$ then $\overbrace{\text{Carol is red}}^{q}$.

$p \to q$

$$\sim(p \to q) \equiv p \wedge \sim q$$

Bob is green and Carol is not red.

End of [Example]

Related Conditional Statements

Definition of the *Converse, the Inverse, and the Contrapositive*

For a statement form ***S***: $p \rightarrow q$

Symbol: *contrapositive(S)*

- **Spoken:** the *contrapositive of S*
- **Meaning:** the statement form: $\sim q \rightarrow \sim p$

Symbol: *converse(S)*

- **Spoken:** the *converse of S*
- **Meaning:** the statement form: $q \rightarrow p$

Symbol: *inverse(S)*

- **Spoken:** the *inverse of S*
- **Meaning:** the statement form: $\sim p \rightarrow \sim q$

[Example]

For the given conditional statement S: If $\overbrace{\text{Bob is green}}^{p}$ then $\overbrace{\text{Carol is red}}^{q}$.

Write the *contrapositive*, the *converse*, and the *inverse*.

Contrapositive: $\sim q \to \sim p$ If carol is not red then Bob is not green.

Converse: $q \to p$ If carol is red then Bob is green.

Inverse $\sim p \to \sim q$ If Bob is not green then Carol is not red.

End of [Example]

Logical equivalences between the Original, Contrapositive, Converse, Inverse, Negation.

Suppose that *S* is the conditional statement form $p \to q$

Make truth tables for *S, contrapositive(S), converse(S), inverse(S), ~S*

Solution:

p	q	~p	~q	S $p \to q$	Contrapositive(S) $\sim q \to \sim p$	Converse(S) $q \to p$	Inverse(S) $\sim p \to \sim q$	~S
T	T	F	F	T	T	T	T	F
T	F	F	T	F	F	T	T	T
F	T	T	F	T	T	F	F	F
F	F	T	T	T	T	T	T	F

these match

these match

these don't match

Observations

- *S* and *Contrapositive(S)* are logically equivalent
- *Converse(S)* and *Inverse(S)* are logically equiv
- *S* and *Converse(S)* are not logically equivalent
- *Inverse(S)* and *~S* are not logically equivalent
- In fact, *~S* is not logically equivalent to any of the others.

Remark:

- The fact that *S* and *Contrapositive(S)* are *logically equivalent* is extremely important in doing proofs. It is often overlooked.
- The fact that *S* and *Converse(S)* are *not* logically equivalent is also important. Many students make the mistake of thinking that *S* and *Converse(S)* are equivalent. They aren't.

Suppose that the $p \to q$ is known to be *true*.

(a)What are values of p, q?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	F

Could be T, T

Could be F, T

Could be F, F

(b) What must be the value of $q \to p$?

Could be $\overset{q}{T} \to \overset{p}{T}$ T

Could be $\overset{q}{T} \to \overset{p}{F}$ F

Could be $\overset{q}{F} \to \overset{p}{F}$ T

Suppose that the $p \to q$ is known to be *false*.

(a)What are values of p, q?

$p \to q$ is only false when p, q are T, F

(b) What must be the value of $q \to p$?

$q \to p$ would be F → T T

Important observation:

Knowing that some conditional statement $p \to q$ is true does not nail down the values of p, q, and so it does not nail down the value of the converse $q \to p$

But knowing that a conditional statement is false does nail down the values of p, q, and so it does nail down the value of the converse $q \to p$

Translating common language into conditional statements

What does *"I will have a cookie if today is Saturday"* mean?

If today is Saturday then I will have a cookie today

Common language: p if q

Logical Form: IF q THEN p

Symbol: $q \rightarrow p$

Translating common language into conditional statements, continued

What does *"I will have an ice cream today only if today is Friday"* mean?

IF I have ice cream today THEN today is friday.

Common language: p only if q

Logical Form: IF p THEN q

Symbol: $p \to q$

Translating *if and only if* into conditional statements

What does Sentence S: *"$\overbrace{\text{I will go Biking today}}^{B}$ if and only if $\overbrace{\text{today is Wednesday}}^{W}$"* mean?

Let B be the phrase *I will go biking today*.

Let W be the phrase *Today is Wednesday*.

Then Sentence S could be abbreviated: B if and only if W

This means (B if W) AND (B only if W)

But that means $(W \to B)$ AND $(B \to W)$

How would the statement form $(W \rightarrow B) \wedge (B \rightarrow W)$ behave?

Investigate with a truth table:

B	W	$B \rightarrow W$	$W \rightarrow B$	$(W \rightarrow B) \wedge (B \rightarrow W)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Observation:

$(W \rightarrow B)$ AND $(B \rightarrow W)$ is true in exactly those rows where B, W have the same truth value.

This brings us to the definition of the biconditional

Definition of the *Biconditional*

symbols: $p \longleftrightarrow q$ $\qquad$ p iff q

spoken: p if and only if q

less commonly spoken: the biconditional of p and q

meaning in symbols: $(p \to q) \wedge (q \to p)$

meaning, in words: (IF p THEN q) AND (IF q THEN p)

clearer meaning, in words: p, q have the same truth value

Translating common language into conditional statements, continued

What does *"p is sufficient for q"* mean?

What does *"p is necessary for q"* mean?

The book explains this quite well in Section 2.2, and you have a couple of exercises about it.

I will leave it to you to read the book discussion of this topic.

End of Video for H02.2