Topic for this Video:

Section 4.3: Direct Proof and Counterexample III: Rational Numbers

In the previous two sections, we had an introduction to proofs using examples that involved only the most basic of defined mathematical terms: *even* and *odd numbers*, *composite* and *prime numbers*, and *consecutive integers*. In today's video, we discuss proofs of statements involving another defined term: *rational numbers*. In the proofs, it will sometimes be useful to use the closely-related defined term *irrational numbers* and to use the *zero product property* of real numbers. The definitions and the statement of the property are below.

Definition of *Rational Number*

Words: r *is a rational number*

Meaning: $(r \in \boldsymbol{R}) \wedge \left(\exists a, b \in \boldsymbol{Z}\left(r = \frac{a}{b}\right)\right)$

Remark: The equation $r = \frac{a}{b}$ involves real number equality. It can only be true if the things on both sides of the equal sign are real numbers. That implies that the denominator $b \neq 0$. That requirement is not explicitly stated in my version of the definition, but it does not need to be because it is implied. (It isexplicitly stated in the book's definition.)

Additional Notation: The symbol $\boldsymbol{Q}$ denotes the set of rational numbers. Note $\boldsymbol{Q} \subset \boldsymbol{R}$.

Definition of *Irrational Number*

Words: r *is an irrational number*

Meaning: r is a real number and r is not rational.

Additional Notation: There is no commonly used symbol for the set of irrational numbers. But the set can be expressed as a set difference: $\boldsymbol{irrationals} = \boldsymbol{R} - \boldsymbol{Q}$.

The *Zero Product Property*

Informal version

If a product of two real numbers is zero, then at least one of the real numbers is zero.

Informal version, contrapositive:

If two real numbers are both non-zero, then their product is non-zero.

Formal version:

$$\forall a, b \in \boldsymbol{R} \left(ab = 0 \rightarrow \big((a = 0) \vee (b = 0) \big) \right)$$

Formal version, contrapositive:

$$\forall a, b \in \boldsymbol{R} \left(\big((a \neq 0) \wedge (b \neq 0) \big) \rightarrow ab \neq 0 \right)$$

[Example 1] Each of the numbers below is rational. Write each as the ratio of two integers.

(a) 5.678

(b) 5.678678 ...

(c) 5.23678678 ...

Solution

(a) $5.678 = \frac{5678}{1000}$

terminating decimal

trick

(b) $5.678678\ldots = r$

repeating decimal

introduce symbol

then $1000r = 1000(5.678678\ldots) = 5678.678\ldots$

So $1000r - r = 5678.\cancel{678678}\ldots - 5.\cancel{678678}\ldots = 5673$

$999r = 5673$

$r = \frac{5673}{999}$

(c) $5.23678678\cdots$

repeating decimal, but the pattern does not start right away.

let $r = 5.23678678\cdots$

then $100r = 523.678678\cdots$

$100{,}000r = 523678.678678\cdots$

So $100{,}000r - 100r = 523678.678678\cdots$

$$- \quad 523.678678\cdots$$

$$99{,}900r = 523155$$

$$r = \frac{523{,}155}{99{,}900}$$

[Example 2] Let S be the statement

The square of any rational number is rational.

(a) Write the formal version of S.

$\forall r \in \mathbb{Q}\ (r^2 \in \mathbb{Q})$

(b) Rewrite the formal version of S in the form of a universal conditional statement.

$\forall x \in \mathbb{R}\ (\text{If } x \in \mathbb{Q} \text{ then } x^2 \in \mathbb{Q})$

(c) Prove statement *S*. $\forall x \in \mathbb{R}$ (If $x \in \mathbb{Q}$ then $x^2 \in \mathbb{Q}$)

Proof (Direct Proof Structure)

(1) Suppose that $x \in \mathbb{R}$ and $x \in \mathbb{Q}$

(2) There exist integers a, b such that $x = \frac{a}{b}$ (by (1) and definition of rational)

this equation tells us that $b \neq 0$

(3) $x^2 = \left(\frac{a}{b}\right)^2$ (by (2))

$= \frac{a^2}{b^2}$

(4) Let $c = a^2$ and let $d = b^2$

Notice that c, d are both integers

And notice that $d \neq 0$ (from (2), we know $b \neq 0$. Zero product property tells us that $b \cdot b \neq 0$)

(5) There exist integers c, d such that $x^2 = \frac{c}{d}$

(6) Therefore $x^2 \in \mathbb{Q}$ (by (5) and definition of rational)

End of proof

(d) Write the formal versions of the *contrapositive*, *converse*, and *inverse* of *S*.

Statement S: $\forall x \in \mathbb{R} \left(\text{If } \overbrace{x \in \mathbb{Q}}^{A} \text{ then } \overbrace{x^2 \in \mathbb{Q}}^{B}\right)$

Contrapositive (S): $\forall x \in \mathbb{R} \left(\text{If } \underbrace{x^2 \notin \mathbb{Q}}_{\sim B} \text{ then } \underbrace{x \notin \mathbb{Q}}_{\sim A}\right)$

Converse (S): $\forall x \in \mathbb{R} \left(\text{If } \underset{B}{x^2 \in \mathbb{Q}} \text{ then } \underset{A}{x \in \mathbb{Q}}\right)$

inverse (S): $\forall x \in \mathbb{R} \left(\text{If } \underset{\sim A}{x \notin \mathbb{Q}} \text{ then } \underset{\sim B}{x^2 \notin \mathbb{Q}}\right)$

(e) Which of the statements in **(d)** are true and which are false? Explain.

Solution

remember that contrapositive(S) $\equiv$ S

we have proven that S is true

So contrapositive(S) is <u>automatically</u> true.

But converse(S) is <u>not</u> logically equivalent to S.

Knowing that S is true does not tell us anything about converse(S)

Consider $x=\sqrt{2}$

Observe $x=\sqrt{2}$ is a real number.

Famous fact (that we will prove in a week or two) $\sqrt{2}$ is <u>irrational</u>

But notice that $x^2=(\sqrt{2})^2=2$ which is <u>rational</u>.

So we have an example of a real number x such that x^2 is rational and x is not rational.

Conclusion: Converse(S) is false

Since Inverse(S) $\equiv$ converse(S),

we see that Inverse(S) is also false.

[Example 3] Let S be the following statement.

The difference of any two rational numbers is rational.

(a) Write the formal version of S.

$\forall r, q \in \mathbb{Q} \; (r - q \in \mathbb{Q})$

(b) Rewrite the formal version of S in the form of a universal conditional statement.

$\forall x, y \in \mathbb{R} \; (\text{If } x, y \in \mathbb{Q} \text{ then } x - y \in \mathbb{Q})$

(c) Prove or disprove statement S. (If S is false, suggest a change that would make it true.)

Statement S: $\forall x, y \in \mathbb{R}$ (If $x, y \in \mathbb{Q}$ then $x - y \in \mathbb{Q}$)

Proof (Direct Proof)

(1) Let $x, y \in \mathbb{R}$ and $x, y \in \mathbb{Q}$ (generic particular elements)

(2) Then there exist integers a, b, c, d such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$ (by (1) and definition of rational)

(3) Observe b, d are not zero (because $\frac{a}{b}$ and $\frac{c}{d}$ are real numbers)

(4) $x - y = \frac{a}{b} - \frac{c}{d}$ (by (2))

$= \frac{ad - bc}{bd}$ (got common denominator)

(5) let $e = ad - bc$ and $f = bd$. Observe e, f are integers.

(6) Observe that $f \neq 0$ (by 3 and the Zero Product Property)

(7) There exist integers e, f such that $x - y = \frac{e}{f}$

(8) Then $x - y \in \mathbb{Q}$ (by (7) and definition of rational)

End of proof.

[Example 4] Let S be the following statement.

The quotient of any two rational numbers is rational.

(a) Write the formal version of S.

$$\forall q, r \in \mathbb{Q} \left(\frac{q}{r} \in \mathbb{Q} \right)$$

(b) Rewrite the formal version of S in the form of a universal conditional statement.

$$\forall x, y \in \mathbb{R} \left(\text{If } x, y \in \mathbb{Q} \text{ then } \frac{x}{y} \in \mathbb{Q} \right)$$

(c) Prove or disprove statement S. (If S is false, suggest a change that would make it true.)

Statement S: $\forall\, x,y \in \mathbb{R}\left(\text{If } x,y \in \mathbb{Q} \text{ then } \frac{x}{y} \in \mathbb{Q}\right)$

explore to see if we think S is true or false.

x, y	$\frac{x}{y}$	
3, 5	$\frac{3}{5}$	rational ✓
5, 3	$\frac{5}{3}$	rational ✓
0, 1	$\frac{0}{1} = 0$	rational ✓
1, 0	$\frac{1}{0}$	undefined!! not rational!!

(also not irrational, because it is not even a real number.

S is false

The example $x=1, y=0$ is a counterexample because x, y are rational but $\frac{x}{y}$ is not rational.

Statement S: $\forall\, x,y \in \mathbb{R} \left(\text{If } x,y \in \mathbb{Q} \text{ then } \frac{x}{y} \in \mathbb{Q} \right)$

Improved Statement

New S: $\forall\, x,y \in \mathbb{R} \left(\text{If } x,y \in \mathbb{Q} \text{ and } y \neq 0 \text{ then } \frac{x}{y} \in \mathbb{Q} \right)$

Think about why this is believable

Suppose $x,y \in \mathbb{Q}$ and $y \neq 0$

So $x = \frac{a}{b}$ and $y = \frac{c}{d}$

we know $b \neq 0$ and $d \neq 0$ because x,y exist as real numbers

we also know that $c \neq 0$ because $y \neq 0$.

So $\frac{x}{y} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

we know $bc \neq 0$ because $b \neq 0$ and $c \neq 0$.