Video for Homework H04.9 The Handshake Theorem

Reading: From Chapter 4 Elementary Number Theorey and Methods of Proof

- Section 4.9 Application: The Handshake Theorem
  - pages 235- – 241, Examples 4.9.1 – 4.9.8

Homework: 4.9#2,3,7,8,10,13,15,16

Topics:

- **The Total Degree of a Graph**
- **The Handshake Theorem**
- **Simple Graphs**
- **Determining whether Certain Simple Graphs Exist**

Recall the definition of *graph* from Section 1.4

Definition of **Graph**

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

And recall the definition of *degree of a vertex* from Section 1.4

Definition of **Degree of a Vertex**

Let G be a graph and v a vertex of G. The **degree of $\boldsymbol{v}$**, denoted $\boldsymbol{deg(v)}$, equals the number of edges that are incident on v, with an edge that is a loop counted twice.

Section 4.9 begins with the definition of *total degree of a graph* and a theorem about it.

Definition of **Total Degree of a Graph**

The **total degree of a graph** is the sum of the degrees of all the vertices of the graph

Theorem 4.9.1 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

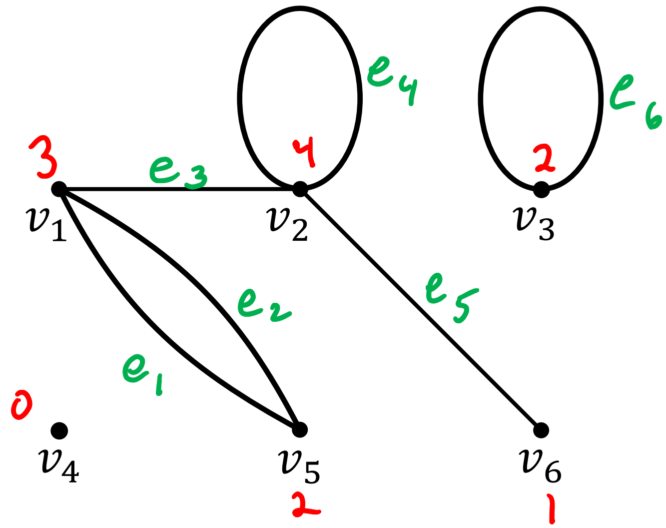
v_2

$$\text{total degree of } G = \deg(v_1) + \deg(v_1) + \cdots + \deg(v_n) = 2 \cdot (\text{number of edges } G)$$

Corollary 4.9.2 The total degree of a graph is even

Proposition 4.9.3 In any graph there is an even number of vertices of odd degree.

[Example 1](similar to 4.9#2) For the graph shown.



(a) Find degree of each vertex

$\deg(v_1) = 3$

$\vdots$

$\deg(v_6) = 1$

(b) Find total degree of the graph.

total degree of $G = 3 + 4 + 2 + 0 + 2 + 1 = 12$

(c) Check that number of edges equals half of the total degree.

6 edges $= \frac{1}{2}\,12 = \frac{\;}{2}$(total degree)

[Example 2] (similar to 4.9#3,7,8)

three oddnumbers

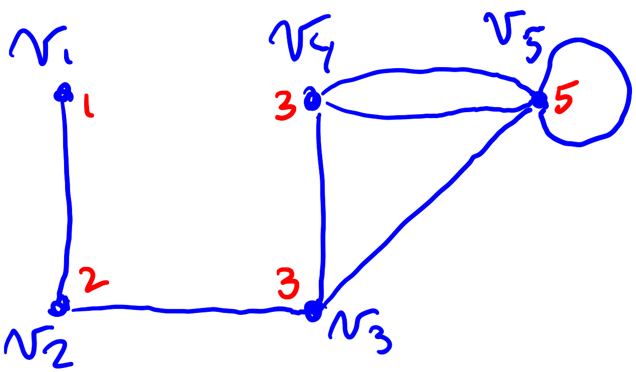
(a) Can a graph with vertices of degrees 0,3,4,5,7 exist?

If not, explain why not.

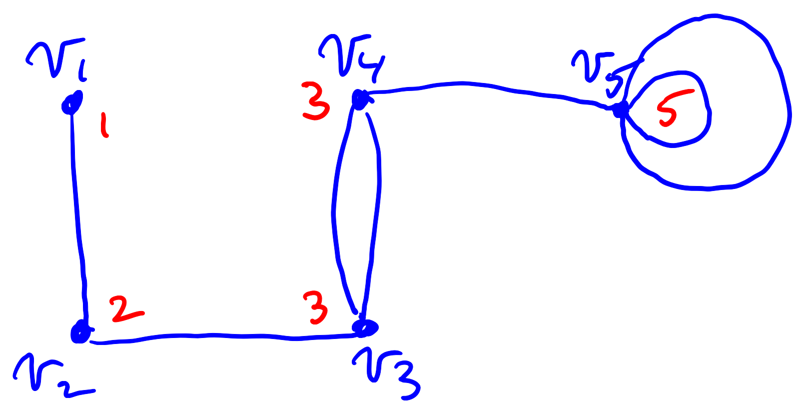
If so, then find total degree, find number of edges, and draw an example

No, Proposition 4.9.3 says that there must be an even number of vertices with odd degree

(b) Same question, but with vertices of degrees 1,2,3,3,5.



one example



another example

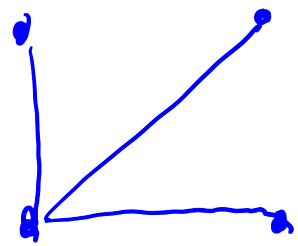
> Definition of **Simple Graph**
>
> A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.

[Example 3] (similar to 4.9#10,13)

(a) Can a simple graph with four vertices of degrees 0,2,2,4 exist?

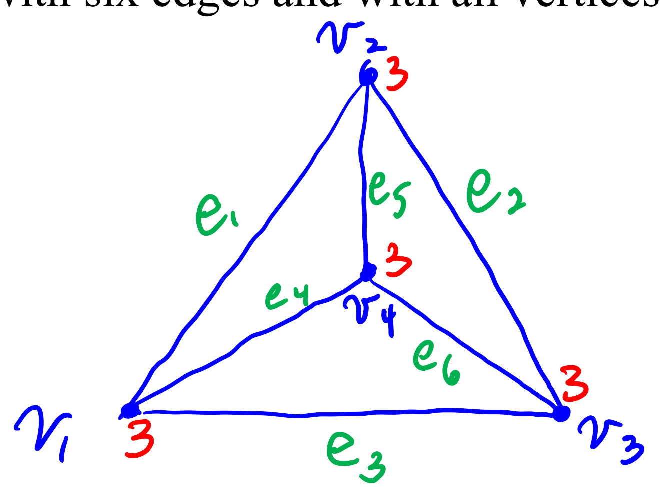
If not, explain. If so, draw an example.

No The max degree possible for a vertex in a simple graph with four vertices would be $\deg(v)=3$



(b) Can a simple graph with six edges and with all vertices of degree 3 exist?

yes!



[Example 4](similar to 4.9#15)

At a party attended by a group of people,

- Two people knew one other person before the party.
- Five people knew two other people before the party.
- The rest of the people knew three other people before the party.
- A total of 15 pairs of people knew each other before the party.

(a) How many people attending the party knew three other people before the party?

Let vertices correspond to people

Let edges connect people who knew each other before party.

15 pairs of people knew each other, So 15 edges,

Therefore, by Handshake Theorem, total degree $= 2 \cdot 15 = 30$

2 vertices have degree 1

5 vertices have degree 2

Let X = number of vertices that have degree 3

total degree $= 30 = 2 \cdot 1 + 5 \cdot 2 + X \cdot 3 = 2 + 10 + 3X$

$30 = 12 + 3X$ So $X = 6$

So six people knew three other people

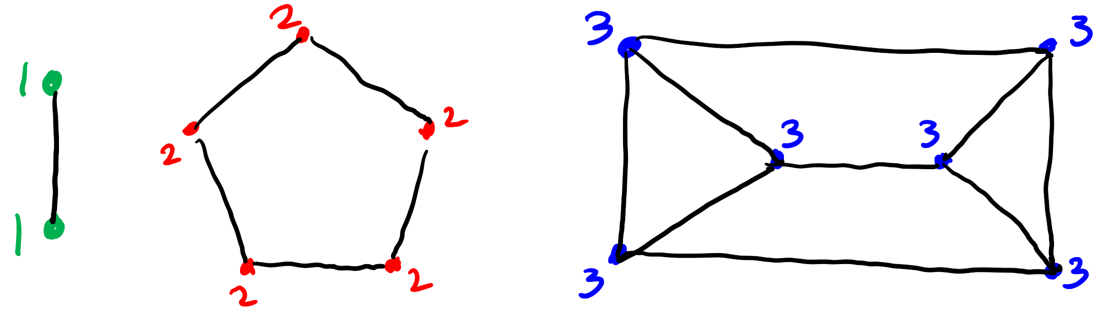
(b) How many people attended the party?

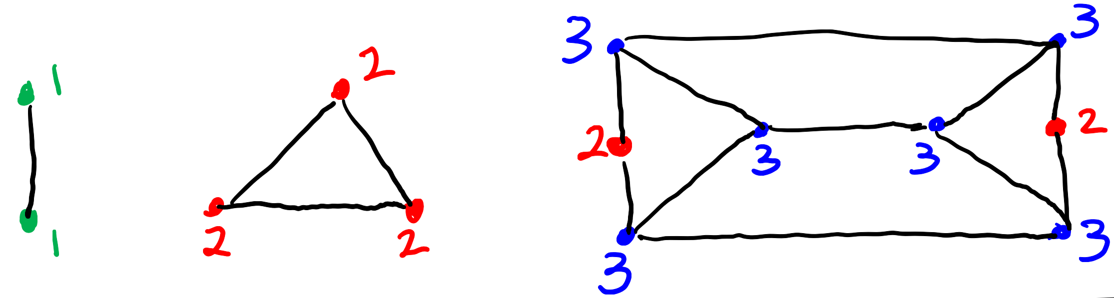
Total # of people = total # of vertices

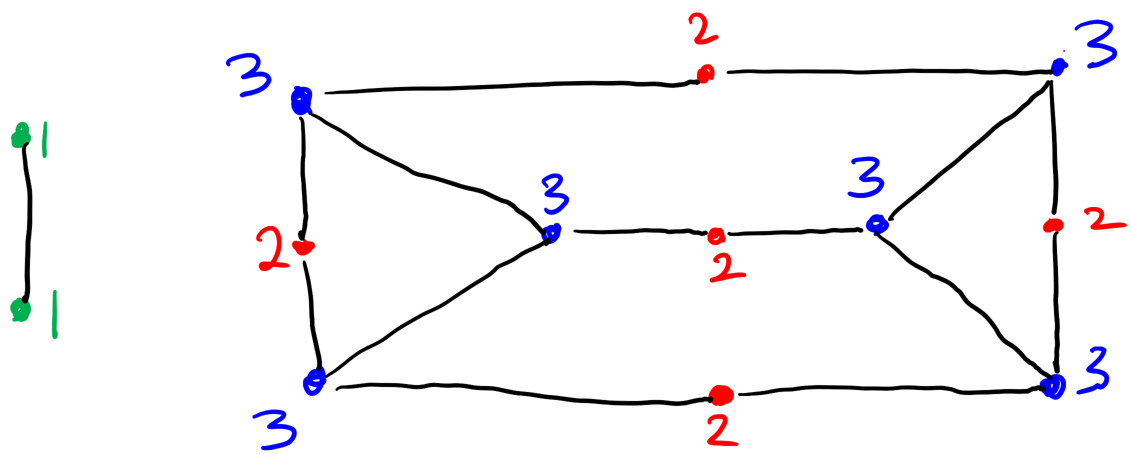
$= 2 + 5 + 6$

$= 13$

(c) Draw three examples of graphs that fit the description.







[Example 5] (similar to 4.9#16)

Assume that friendship is a symmetric relationship: If x is a friend of y, then y is a friend of x.

(a) In a group of 5 people, is it possible for each person to have exactly 3 friends?

Let vertices correspond to people in the group. So 5 vertices.
Let edges connect edges corresponding to people that are friends.
So the number of edges that touch a vertex = number of friends that person has.
That is degree of vertex = number of friends.
Question becomes: Can a graph have vertices with degrees 3, 3, 3, 3, 3?
Impossible. Must have an even number of vertices of odd degree.

(b) In a group of 6 people, is it possible for each person to have exactly 3 friends? Explain.