Video for Homework H06.1 Set Theory: Definitions and the Element Method of Proof

Set Theory is a topic in Discrete Math that you have some familiarity with. The concepts of sets, elements, subsets, unions, complements, etc, are all concepts that you have seen before. But there is profound subtlety in the subject. You might find that you become confused about things that you did not previously find confusing.

To start with, it is important to realize that the words *set* and *element* are undefined terms in set theory. You may think that you know what they mean, but you don't really.

For instance, you might claim that the word *set*, actually has a definition, such as this one:

"A *set* is a collection of things called *elements*."

But then, what is a *collection*? One must either provide a definition of *collection*, or just admit that the word *collection* is undefined. In that case, the proposed definition of the word *set* is really just saying that the words *set* and *collection* are declared to mean the same thing, but what that thing is, is…undefined. Confused yet?

It is therefore important, when studying set theory, to be as clear as possible about what terms are *undefined* and what terms are *defined*. For our purposes in MATH 3050, we won't need to deal with the idea of undefined terms beyond the confusing stuff that I said on the previous page. But be aware that the subject of Set Theory is more subtle than you realize.

I will present definitions in a different order from the book, in order to be able to give more precise definitions for set terminology than the definitions given in the book.

The Universal Set

The author introduces the idea of a *Universal Set* a few pages into Section 6.1. It is useful to discuss it first, because it allows us to make more precise definitions of Set Theory terminology. Here is what the author says about the *Universal Set* on page 381

> Most mathematical discussions are carried on within some context. For example, in a certain situation all sets being considered might be sets of real numbers. In such a situation, the set of real numbers would be called a **universal set** or a **universe of discourse** for the discussion.

It is useful to have a common symbol for the Universal Set.

The ***Universal Set*** will be denoted by the letter U.

Set Equality

It is common for authors to define *set equality* in terms of *subsets*. The author of our book does this, on page 379. But I feel that it is very important to introduce the definition of set equality *before* the definition of subset, in order to realize that set equality is a simpler, more fundamental concept. Here's my definition.

Set Equality (simplest definition)

Symbol: $A = B$

Spoken: A equals B

Usage: A and B are sets

Meaning: Sets A and B have the same elements. Another way of saying this is that regardless of the element x chosen from the universal set U, the statements $x \in A$ and $x \in B$ will *both* be *true* or they will *both* be *false*.

Subsets

With the prior introduction of the concepts of the Universal Set and Set Equality, it is possible to make precise, clear definitions of the terms *Subset* and *Proper Subset*.

Definition of Subset

Symbol: $A \subseteq B$

Spoken: *A is a **subset** of B*

Usage: A and B are sets

Meaning

Stated as a Universal Statement: $\forall x \in A(x \in B)$

Stated as a Universal Conditional Statement: $\forall x \in U(\text{If } x \in A \text{ then } x \in B)$

Proper Subsets

Definition of ProperSubset

Symbol: $A \subsetneq B$

$A \subsetneq B$

Spoken: *A is a **proper subset** of B*

Usage: A and B are sets

Meaning

Meaning in words: A is a subset of B and A is not equal to B.

Meaning in symbols: $(A \subseteq B) \wedge (A \neq B)$

Remark on the notation: It would be nice if the symbols $\subseteq$ and $\subset$ could be used for subset and proper subset, analogous to the inequality symbols $\leq$ and $<$. But there is a problem in that some books use the symbol $\subset$ to denote a subset, not a proper subset. Because there is possibility of confusion, there is really no choice but to simply not use the symbol $\subset$. I will use $\subseteq$ and $\subsetneq$ to denote subset and proper subset in these notes. (Notice that the textbook does not introduce a symbol for proper subset.)

Proving or disproving statements that use the terminology of subsets.

What does it mean to say that *A* is *not* a subset of *B*?

$$\sim(A \subseteq B) \equiv \sim(\forall x \in U(\text{If } x \in A \text{ then } x \in B))$$

$$\equiv \exists x \in U(\sim(\text{If } x \in A \text{ then } x \in B))$$

$$\equiv \exists x \in U(x \in A \text{ and } \sim(x \in B))$$

$$\equiv \exists x \in U(x \in A \text{ and } x \notin B)$$

[Example 1] (similar to 6.1#4)

Let $A = \{n \in \mathbf{Z} | n = 6r - 5 \;\; for\ some\ integer\ r\}$

Let $B = \{m \in \mathbf{Z} | m = 3s + 1 \;\; for\ some\ integer\ s\}$

(a) Prove or disprove: $A \subseteq B$

Solution

Explore. See what is in A and what is in B

examples of $n = 6r - 5$:

$n = 6(-1) - 5 = -11$

$n = 6(0) - 5 = -5$

$n = 6(1) - 5 = 6 - 5 = 1$

$n = 6(2) - 5 = 12 - 5 = 7$

$n = 6(3) - 5 = 18 - 5 = 13$

$\vdots$

$A = \{\ldots, -11, -5, 1, 7, 13, \ldots\}$

$B = \{\ldots, -5, -2, 1, 4, 7, 10, 13, \ldots\}$

It seems that $A \subseteq B$ is true.

We prove that $A \subseteq B$ using the Element Method of Proof

We want to prove that $A \subseteq B$

That is, prove that $\forall x \in U$ (If $x \in A$ then $x \in B$)

Proof (Direct Proof)

(1) Suppose $x \in U$ and $x \in A$ (generic particular element)

(2) Then $x = 6k - 5$ for some integer k (by (1) and definition of set A)

trick

(3) then $x = 6k - 6 + 1$

$= 6(k-1) + 1$

$= 3 \cdot \underbrace{2 \cdot (k-1)} + 1$

(4) Let $m = 2(k-1)$ Then m is an integer and $x = 3m+1$

(5) $x = 3m+1$ for some integer m

(6) Therefore $x \in B$ {by (5) and definition of set B}

End of proof

(b) Prove or disprove: $B \subseteq A$

We see that B is <u>not</u> a subset of A.

So we want to <u>disprove</u> the statement $B \subseteq A$

$B \subseteq A$: $\forall x \in U$ (If $x \in B$ then $x \in A$)

$B \notin A$: $\exists x \in U$ ($x \in B$ and $x \notin A$)

Example: let $x = 4$. Then $x \in B$ and $x \notin A$.

Alternate Presentation of the Definition of Set Equality

Returning to the definition of set equality, we can add a formal presentation of the definition, and we see that the formal presentation can be shortened by using the terminology of *subsets*.

Set Equality (definition using subset terminology and notation)

Symbol: $A = B$

Spoken: A equals B

Usage: A and B are sets

Meaning: Sets A and B have the same elements. Another way of saying this is that regardless of the element x chosen from the universal set U, the statements $x \in A$ and $x \in B$ will both be true or they will both be false.

Meaning stated formally: $\forall x \in U\big((If\ x \in A\ \ then\ x \in B) \wedge (If\ x \in B\ \ then\ \underline{x \in A})\big)$

Meaning stated formally using Subset Notation: $A \subseteq B\ and\ B \subseteq A$

Operations on Sets (from p. 381)

Symbol: $A \cup B$

Spoken: *The union of A and B*

Meaning: $\{x \in U | x \in A \text{ or } x \in B\}$

Symbol: $A \cap B$

Spoken: *The intersection of A and B*

Meaning: $\{x \in U | x \in A \text{ and } x \in B\}$

Symbol: A^c

Spoken: *The complement of A*

Meaning: $\{x \in U | x \notin A\}$

Symbol: $B - A$

Spoken: *B minus A*

Meaning: $\{x \in U | x \in B \text{ and } x \notin A\}$

More abbreviated meaning: $B \cap A^c$

[Example 2] (similar to 6.1#35) Let $A = \{1\}, B = \{u, v\}, C = \{m, n\}$

(a) Find $A \cup (B \times C)$

$B \times C$, B cross C, the cartesian product of B and C

$$B \times C = \{(b, c) \mid b \in B \text{ and } c \in C\}$$

$$= \{(u,m), (u,n), (v,m), (v,n)\}$$

$$A \cup (B \times C) = \{1, (u,m), (u,n), (v,m), (v,n)\}$$

(b) Find $(A \cup B) \times C$

$$A \cup B = \{1\} \cup \{u, v\} = \{1, u, v\}$$

$$(A \cup B) \times C = \{1, u, v\} \times \{m, n\}$$

$$= \{(1, m), (u, m), (v, m), (1, n), (u, n), (v, n)\}$$

Complements of Unions, Intersections, and Differences

To determine the *Complements* of *Unions*, *Intersections*, and *Differences*, we have to use the formal presentation of the meanings of those sets. Negations of and/or statements will be involved, which means that De Morgan's laws will be needed.

[Example 3] (similar to 6.1#8,9) find $(A \cap B)^c$

$(A \cap B)^c = \{x \in U \mid x \notin (A \cap B)\}$ definition of complement

$= \{x \in U \mid \sim(x \in (A \cap B))\}$ meaning of $\notin$

$= \{x \in U \mid \sim((x \in A) \text{ and } (x \in B))\}$ definition of $\cap$

$= \{x \in U \mid \sim(x \in A) \text{ or } \sim(x \in B)\}$ DeMorgan

$= \{x \in U \mid x \notin A \text{ or } x \notin B\}$ meaning of $\notin$

$= \{x \in U \mid x \in A^c \text{ or } x \in B^c\}$ meaning of complement

$= A^c \cup B^c$ Definition of union

The Empty Set

Definition of the Empty Set

Symbol: ϕ phi

Spoken: the empty set

Meaning: The set with no elements

[Example 4] (similar to 6.1#18)

(a) Is the number 0 in ϕ? Why? No. The empty set has no elements.

(b) Is $\phi = \{\phi\}$? No. ϕ has no elements. $\{\phi\}$ has one element. Those two things can't be equal.

(c) Is $\phi \in \{\phi\}$? True

(d) Is $\phi \in \phi$? No. The empty set has no elements

(e) (The following question is about a fact that you may have heard before, but may have not understood and not known how to prove.)

Prove that the empty set is a subset of every set.

Statement S

Written formally: $S: \forall A \subseteq U (\phi \subseteq A)$

If S is false, then $\sim S$ would be true

$\sim S \equiv \sim(\forall A \subseteq U(\phi \subseteq A))$

$\equiv \exists A \subseteq U(\sim(\phi \subseteq A))$

$\equiv \exists A \subseteq U(\exists x \in U(x \in \phi \text{ and } x \notin A))$

To prove $\sim S$ is true, we would need an example of an $x \in \phi$ that is not an element of A. But ϕ has no elements!

So $\sim S$ cannot be true. Conclude that S is true.

(f) Is $\phi \subseteq \phi$?

Yes. We just proved this in (e)

Disjoint Sets, Mutually Disjoint Sets, Partitions of Sets

Definition

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \varnothing.$$

Definition

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \varnothing \quad \text{whenever } i \neq j.$$

Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

[Example 5] (similar to 6.1#28,29,30)

(a) Is $\{\boldsymbol{R}^+, \boldsymbol{R}^-\}$ a partition of $\boldsymbol{R}$? Explain.

observe: $\mathbb{R}^+, \mathbb{R}^-$ are disjoint ✓

But $\mathbb{R}^+ \cup \mathbb{R}^- \neq \mathbb{R}$ because $0 \notin \mathbb{R}^+ \cup \mathbb{R}^-$ ✗

So $\mathbb{R}^+, \mathbb{R}^-$ is not a partition

(b) Is $\{\boldsymbol{Z}, \boldsymbol{Q}\}$ a partition of $\boldsymbol{R}$? Explain.

No! $\mathbb{Z}, \mathbb{Q}$ are not disjoint. ✗

For instance $5 \in \mathbb{Z}$ and $5 \in \mathbb{Q}$ because $5 = \frac{5}{1}$

and $\mathbb{Z} \cup \mathbb{Q} \neq \mathbb{R}$ ✗

because $\mathbb{R}$ contains irrational numbers that are not in $\mathbb{Z} \cup \mathbb{Q}$

(c) Is $\{even\ integers, odd\ integers\}$ a partition of $\boldsymbol{Z}$? Explain.

yes: even integers $= \{n = 2k$ for some integer $k\}$

odd integers $= \{n = 2k+1$ for some integer $k\}$

The quotient remainder theorem with $d=2$ tells us that every integer n can be written as either $n=2k$ or $n=2k+1$.

(d) Is $\{prime\ numbers, composite\ numbers\}$ a partition of $\boldsymbol{Z}$? Explain.

Observe: prime & composite are disjoint, but their union is not all of the integers.

The numbers $-5, 0, 1$ are neither prime nor composite.

Not a partition!

Unions and Intersections of an Indexed Collection of Sets

Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for every } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for every nonnegative integer } i\}.$$

[Example 6] (similar to 6.1#23) For each non-negative integer i, define the set $D_i = [-i, i]$

$(\boldsymbol{a})$ $Find$ $\bigcup_{i=1}^{4} D_i$ $= D_1 \cup D_2 \cup D_3 \cup D_4$

$= [-1,1] \cup [-2,2] \cup [-3,3] \cup [-4,4]$

$= [-4,4]$

$= D_4$

$(\boldsymbol{b})$ $Find$ $\bigcap_{i=1}^{4} D_i$ $= D_1 \cap D_2 \cap D_3 \cap D_4$

$= [-1,1] \cap [-2,2] \cap [-3,3] \cap [-4,4]$

$= [-1,1]$

$= D_1$

(c) Are $D_0, D_1, D_2, \ldots$ mutually disjoint? Explain

no All those sets contain the number 0

$D_0 = [0,0] = 0 \le x \le 0 = \{0\}$

(**d**) *Find* $\bigcup_{i=1}^{n} D_i$ $= D_n = [-n, n]$

(**e**) *Find* $\bigcap_{i=1}^{n} D_i$ $= D_1 = [-1, 1]$

(*f*) Find $\bigcup_{i=1}^{\infty} D_i = \mathbb{R}$

(*g*) Find $\bigcap_{i=1}^{\infty} D_i = D_1 = [-1, 1]$

(h) Find $\bigcap_{i=0}^{\infty} D_i = \{0\} \cap [-1, 1] \cap [-2, 2] \cap \dots$

$= \{0\}$

$= D_0$