Video for Homework H06.2 Properties of Sets

Investigating Statements about Sets in Two Ways

1) **Visualizing them using Venn Diagrams**
2) **Proving them using an *Element Argument***

Recognizing the *Contrapositive* to make a proof simple

Using an *Element Argument* to prove that a set is the empty set.

Proving Statements About Sets Using Theorems and Previously Proved Results

Some really old stuff that will be useful:

Chapter 2 Theorem about Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6.	*Double negative law:*	$\sim(\sim p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

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Chapter 2 Table of Valid Argument Forms

TABLE 2.3.1 Valid Argument Forms

Modus Ponens	$p \rightarrow q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \rightarrow q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \rightarrow \mathbf{c}$ $\therefore p$

More recent stuff (From Section 6.1) that will be useful.

Definition of Subset

Symbol: $A \subseteq B$

Spoken: *A is a* ***subset*** *of B*

Usage: A and B are sets

Meaning

Stated as a Universal Statement: $\forall x \in A (x \in B)$

Stated as a Universal Conditional Statement: $\forall x \in U (\text{If } x \in A \text{ then } x \in B)$

Set Equality (definition using subset terminology and notation)

Symbol: $A = B$

Spoken: A equals B

Usage: A and B are sets

Meaning: Sets A and B have the same elements. Another way of saying this is that regardless of the element x chosen from the universal set U, the statements $x \in A$ and $x \in B$ will both be true or they will both be false.

Meaning stated formally using Subset Notation: $A \subseteq B \ and \ B \subseteq A$

Operations on Sets (from p. 381)

Symbol: $A \cup B$

Spoken: *The union of A and B*

Meaning: $\{x \in U | x \in A \text{ or } x \in B\}$

Symbol: $A \cap B$

Spoken: *The intersection of A and B*

Meaning: $\{x \in U | x \in A \text{ and } x \in B\}$

Symbol: A^c

Spoken: *The complement of A*

Meaning: $\{x \in U | x \notin A\}$

Symbol: $B - A$

Spoken: *B minus A*

Meaning: $\{x \in U | x \in B \text{ and } x \notin A\}$

More abbreviated meaning: $B \cap A^c$

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Investigating Statements about Sets in Two Ways

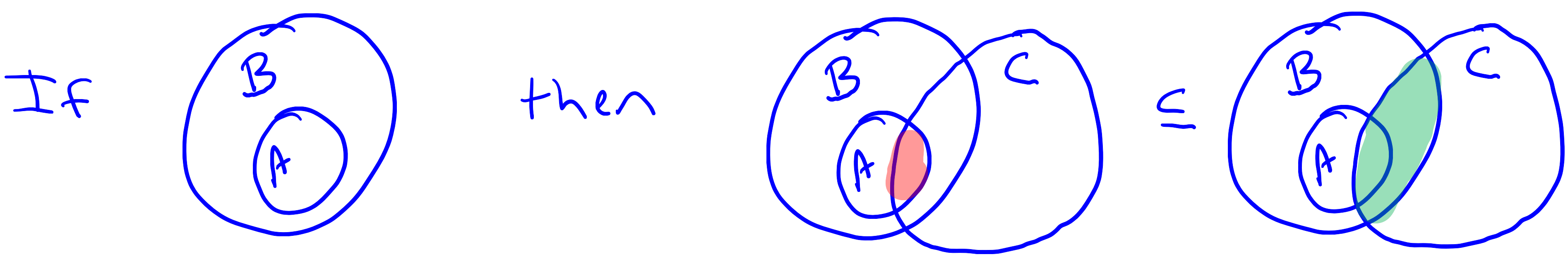
1) **Visualizing them using Venn Diagrams**
2) **Proving them using an *Element Argument***

[Example 1] 6.2#16

Consider the following ~~Set Identity:~~ Statement about sets

For all sets A, B, C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$

(a) Draw Venn Diagrams to illustrate the statement.



If $A \subseteq B$ then $A \cap C \subseteq B \cap C$

(b) Prove the Statement using an *Element Argument* $\forall A,B,C\,(\text{If } A \subseteq B \text{ then } A \cap C \subseteq B \cap C)$

Proof (Direct Proof)

(1) Suppose sets A, B, C are given and $A \subseteq B$ (generic particular element)

(2) For all $x \in A$ $(x \in B)$ (by (1) and definition of subset)

(3) Suppose $y \in A \cap C$ (generic particular element)

(4) Then $y \in A$ and $y \in C$ (by (3) and definition of intersection)

(5) $y \in A$ by (4) and specialization

(6) $y \in C$ by (4) and specialization

(7) $y \in B$ (by (5) and (2))

(8) therefore $y \in B$ and $y \in C$ (by conjunction of (6), (7))

(9) therefore $y \in B \cap C$ (by (8) and definition of subset

(10) For all $y \in (A \cap C)\,(y \in B \cap C)$ (by (3), (9))

(11) Therefore $A \cap C \subseteq B \cap C$ (by (11) and definition of subset)

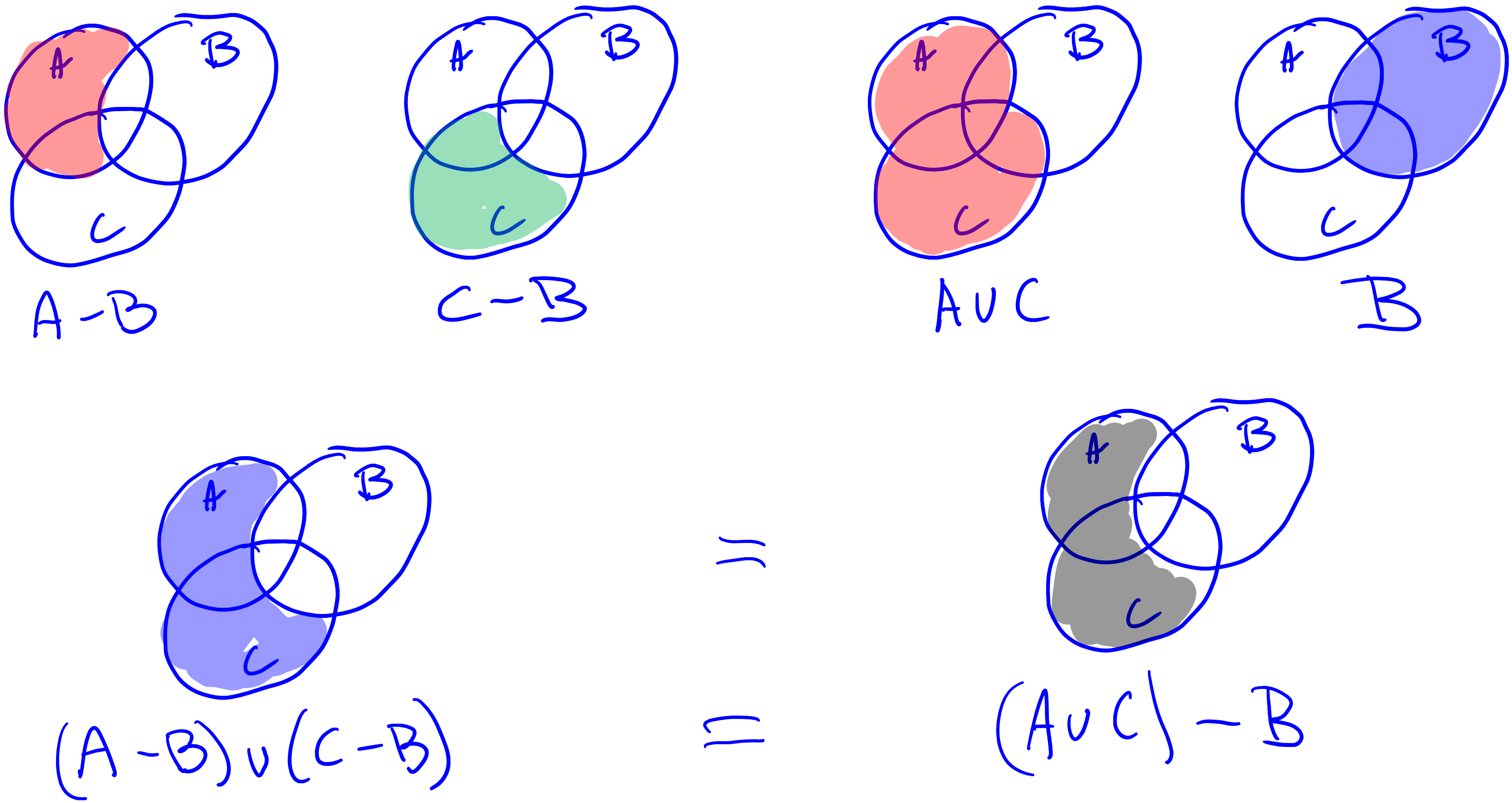
End of proof

[Example 2] 6.2#9 (similar to Homework Problem 6.2#13)

Consider the following Set Identity:

For all sets A, B, C, $(A - B) \cup (C - B) = (A \cup C) - B$

(a) Draw Venn Diagrams to illustrate the set identity.



(b) Prove the Set Identity using an *Element Argument*

$$\forall A,B,C\ \left((A-B)\cup(C-B) = (A\cup C)-B \right)$$

Proof

(1) Suppose Sets A, B, C are given (generic particular element)

Part 1 Prove $(A-B)\cup(C-B) \subseteq (A\cup C)-B$

(2) suppose $x \in ((A-B)\cup(C-B))$

(3) $x \in A-B$ or $x \in C-B$ (by (2) and definition of union)

(4) $x \in A\cap B^c$ or $x \in C\cap B^c$ (by (3) and definition of difference)

(5) $(x\in A$ and $x \in B^c)$ or $(x \in C$ and $x\in B^c)$ (by (4) and definition of intersection)

(6) $(x\in A$ or $x\in C)$ and $x\in B^c$ (by (5) and distributive law

(7) $(x \in (A\cup C))$ and $x \in B^c$ (by (6) and definition of union)

(8) $x \in (A\cup C)\cap B^c$ (by (7) and definition of intersection)

(9) therefore $x \in (A\cup C)-B$

(10) Conclude $(A-B)\cup(C-B) \subseteq (A\cup C)-B$

End of Proof part 1

Part 2 Prove that $(A \cup C) - B \subseteq (A-B) \cup (C-B)$

Similar details

Therefore $(A \cup C) - B \subseteq (A-B) \cup (C-B)$

End of proof Part 2

(*) Therefore $(A-B) \cup (C-B) = (A \cup C) - B$

End of proof

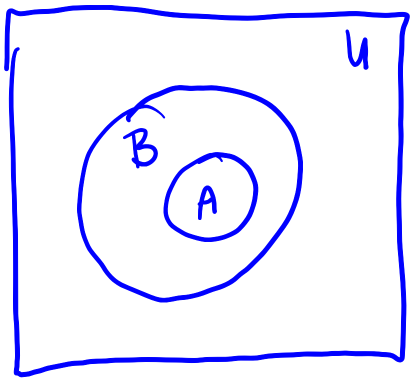
[Example 3] 6.2#18

Consider the following statement about sets

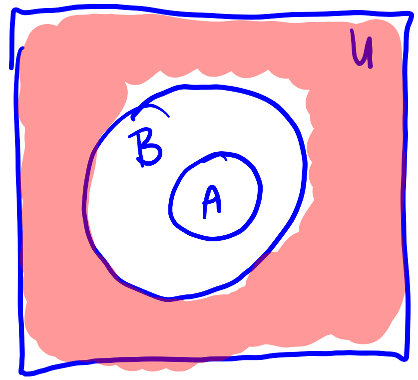
For all sets A, B If $A \subseteq B$ then $B^c \subseteq A^c$

(a) Draw Venn Diagrams to illustrate the statement.

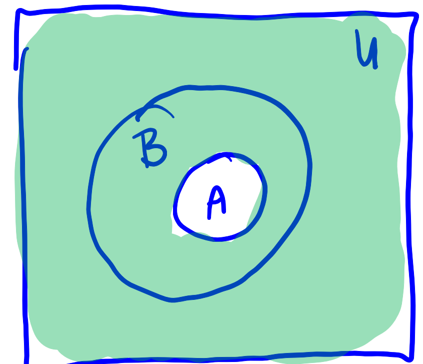
If



then



$\subseteq$



If $A \subseteq B$ then $B^c \subseteq A^c$

(Recognizing the *Contrapositive* to Make a Proof Simple)

(b) Prove the Statement using an *Element Argument* Prove $\forall A,B$ (If $A \subseteq B$ then $B^c \subseteq A^c$)

Proof (Direct Proo

(1) Suppose Sets A,B are given and that $A \subseteq B$ (generic particular element)

(2) $\forall x \in U$ (If $x \in A$ then $x \in B$) (by (1) and definition of subset)

(3) $\forall y \in U$ (If $x \notin B$ then $x \notin A$) contrapositive of (2)

(4) $\forall y \in U$ (If $y \in B^c$ then $y \in A^c$) by (3) and definition of complement)

(5) Therefore $B^c \subseteq A^c$ (by (4) and definition of subset)

End of proof

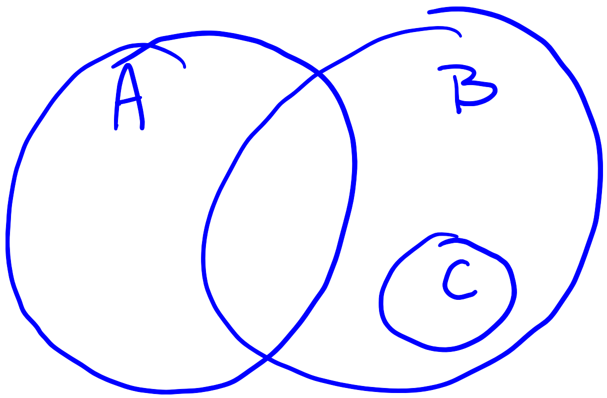
[Example 4] 6.2#36

Consider the following statement about sets:

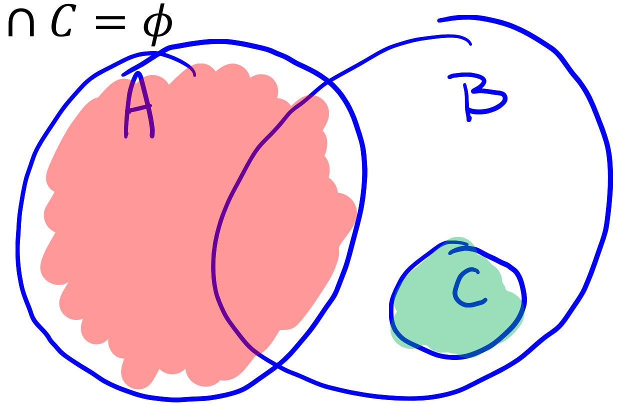
For all sets A, B, C If $C \subseteq B - A$ then $A \cap C = \phi$

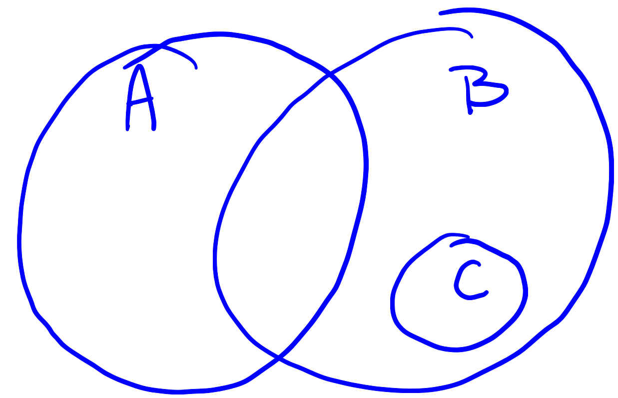
(a) Draw a Venn Diagram to illustrate the statement.

If



then





$A \cap C$

If $C \subseteq B - A$ then $A \cap C = \phi$

(Using an *Element Argument* to prove that a set is the empty set)

(b) (similar to homework problem 6.2#35) Prove the statement using an *Element Argument*

Statement S: $\forall A, B, C\ (\text{If } C \subseteq B - A \text{ then } A \cap C = \emptyset)$

The negation

$\sim S$: $\exists A, B, C\ (C \subseteq B - A \text{ and } A \cap C \neq \emptyset)$

Proof of S (by contradiction)

(1) Assume that S is false

So $\exists A, B, C\ (C \subseteq B - A \text{ and } A \cap C \neq \emptyset)$

(2) Since $A \cap C \neq \emptyset$, there exists an $x \in A \cap C$

(3) $x \in A$ and $x \in C$ (by (2) and definition of intersection)

(4) $x \in A$

(5) $x \in C$

by 3 and specialization

(6) $x \in B - A$ because $C \subseteq B - A$

(7) $x \in B \cap A^c$ by (6) and definition of difference

(8) $x \in B$ and $x \in A^c$ by (7) and definition of $\cap$

(9) $x \in A^c$ by (8) and specialization

(10) $x \notin A$ by (9) and definition of complement

(11) We have reached a contradiction ((10) contradicts (4))

So our assumption in (1) was wrong.

S cannot be false

Therefore, S is true

End of proof.

Proving Statements About Sets Using Theorems and Previously Proved Results

The Element Argument that we used in **[Example 2]** to prove the Set Identity

$$\text{For all sets } A, B, C,\ \ (A - B) \cup (C - B) = (A \cup C) - B$$

is the same sort of argument that would be used to prove two theorems presented in Section 6.2.

Section 6.2 Theorem about Subset Relations

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Section 6.2 Theorem about Set Identities

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For every set A,

$$\text{(a) } A \cup \varnothing = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:* For every set A,

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \varnothing.$$

6. *Double Complement Law:* For every set A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For every set A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For every set A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \varnothing = \varnothing.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\varnothing$:*

$$\text{(a) } U^c = \varnothing \quad \text{and} \quad \text{(b) } \varnothing^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

The idea of stating (and proving) these theorems is that they give us tools that can be used to prove *other subset relations* and *other set identities*. In particular, those theorems give us another way of proving the set identity that we proved in **[Example 2]** and **[Example 4]**. We will revisit those examples and do that kind of proof now.

[Example 2] revisited

Continuing our discussion of the set identity:

For all sets A, B, C, $(A - B) \cup (C - B) = (A \cup C) - B$

(c) Prove the *set identity again*, but this time, instead of using an element argument, use *known Set Identities* from *Theorem 6.2.2*

Proof

(1) $(A - B) \cup (C - B) = (A \cap B^c) \cup (C \cap B^c)$

th 6.2.2.12

$= (A \cup C) \cap B^c$

Th 6.2.2.3b distributive law

$= (A \cup C) - B$

Th. 6.2.2.12

[Example 4] revisited

Continuing our discussion of the statement about sets:

For all sets A, B, C If $C \subseteq B - A$ then $A \cap C = \phi$

(c) Prove the *set identity again*, but this time, instead of using an element argument, use *Theorems* and *previously proved results*

Proof

(1) Suppose A, B, C satisfy $C \subseteq B - A$ (generic particular elements)

(2) $A \cap C \underset{\uparrow}{\subseteq} A \cap (B - A) \underset{\uparrow}{=} A \cap (\underbrace{B \cap A^c}) \underset{\uparrow}{=} A \cap (\underbrace{A^c \cap B}) \underset{\uparrow}{=} (A \cap A^c) \cap B$

using [Example 1] and fact that $C \subseteq B - A$

Th 6.2.2.12

Th 6.2.2.1

Th 6.2.2.2a

$\underset{\uparrow}{=} \phi \cap B \underset{\uparrow}{=} \phi$

Th 6.6.2.5b

th. 6.2.2.8(b)

We have shown $A \cap C \subseteq \phi$

(3) From previous video, $\phi \subseteq A \cap C$

(the empty set is a subset of every set)

(4) Therefore $A \cap C = \phi$ by (2), (3) and the definition of subset.

End of Proof