Video for Homework H07.1

Reading: Section 7.1 Functions in Susanna Epp's book Discrete Mathematics

Homework: H07.1: 7.1#5,6,7,12,14,18,25,28,32,39,42

Topics:

- **Definition of Function**
- **Examples of Functions**
- **Function Equality**
- **Images of Sets and Preimages of Sets**

Recall from Chapter 1:

Ordered Pairs and *Ordered n-tuples*, definitions from Chapter 1

Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$**, consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1$, $x_2 = y_2, \ldots,$ and $x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

The *Cartesian Product* of Sets, definition from Chapter 1

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Exponent notation for some Cartesian Products

Just as the cartesian product $\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$ is often denoted $\boldsymbol{R}^{\mathbf{3}}$, one can denote the Cartesian product of n copies of any set

$$\underbrace{A \times A \times \cdots \times A}_{n}$$

by the symbol

$$A^n$$

In particular, we can denote the Cartesian product

$$\underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n}$$

by the symbol

$$\{0,1\}^n$$

Relations and Functions, definitions from Chapter 1

Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$x\ R\ y \quad \text{means that} \quad (x, y) \in R.$$

The notation $x \not R\ y$ means that x is not related to y by R:

$$x \not R\ y \quad \text{means that} \quad (x, y) \notin R.$$

Definition

A **function F from a set A to a set B** is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

$f: A \to B$

$y = f(x)$

.

In Chapter 7, we study functions in more detail. We begin with a more complete definition of function, one that includes some associated terminology. Here is the Chapter 7 Definition

Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain** of f, to Y, the **co-domain** of f, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically:

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. When x is an element such that $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically:

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Examples of functions

Particularly Simple Function: The *Identity Function*

The Identity Function on a Set

Symbol: I_X or id_X

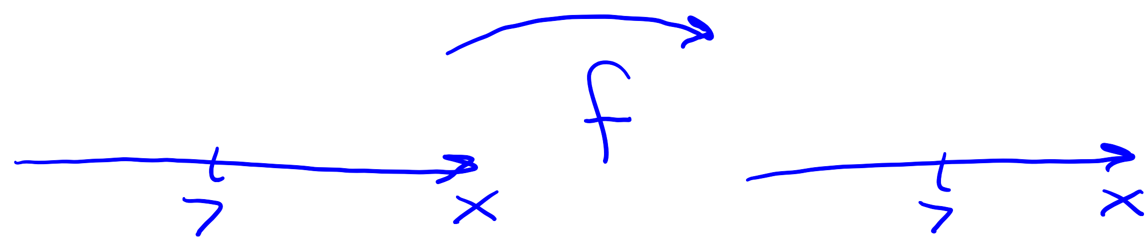
Spoken: the *identity function* on X

Usage: X is a set

Meaning : a function $I_X: X \to X$ defined by $I_X(x) = x$

[Example 1] (similar to 7.1#5)

$(\boldsymbol{a})$ $I_{\boldsymbol{R}}(7) =$ 7



$(\boldsymbol{b})$ $I_{\boldsymbol{R}}\left(e^{(-\sin(\theta))}\right) =$ $e^{(-\sin(\theta))}$

Sequences can be thought of as functions

In Section 5.1, we discussed sequences. In the video for H05.1, I introduced a simple definition of sequence as a list of numbers.

Definition of Sequence (from video for H05.1)

A ***sequence*** is a list of numbers

If the list ends, the sequence is called a ***finite sequence***.

If the list goes on forever, the sequence is called an ***infinite sequence***.

The numbers on the list are called the ***terms*** of the sequence.

The first term of the sequence is called the ***initial term***.

If the last term of a finite sequence is called the ***final term***.

Two sequences are said to be the same if they are the same list of numbers.

But in Section 5.1, we also found explicit formulas for lists. In other words, a *sequence* can also be thought of as a *function*. Here is the book's definition of sequence, from Section 5.1.

Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

Notice that the definition allows for lists of things that are not numbers. For instance, the list of months of the year could be thought of as a sequence whose domain is the set $\{1,2,\ldots,12\}$.

In our Examples from the video for H05.1, we saw that the form of the explicit formula depends on the choice of the starting index.

[Example 2] (From Video for H05.1) Consider the sequence $3,6,12,24,48,96$

(a) Find an explicit formula for the sequence, using a starting index of 0.

(b) Find an explicit formula for the sequence, using a starting index of 1.

In the terminology of functions, we would say that the formula for the function depends on the choice of *domain*. That is, the questions from the example above could be rewritten:

[Example 2] (revisited)(similar to 7.1#6) Consider the sequence $3,6,12,24,48,96$

(a) Find a function with domain the set Z^{nonneg} that describes the sequence.

(b) Find a function with domain the set Z^+ that describes the sequence.

You have a homework problem of this sort. (7.1#6)

Functions with More General Domains

Most (or all) of your previous experience with functions has been with functions whose domains are sets of numbers. But nothing in the definition of function requires that the domain be a set of numbers.

Often it is useful to use functions that take as input a *set*, rather than a *number*. And in those situations, one is interested in investigating a particular collection of sets of a certain type. It is helpful to have terminology that narrows down the category of sets that are being considered, and to have notation for that category. One term that is useful is the power set of a given set.

> **Definition**
>
> Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

In these notes, the font that I will use for the power set is a slightly different script:

$$\mathcal{P}(A)$$

[Example 3] (simlar to 7.1#7) Consider the following set of people

$$A = \{\text{Ann,Bill,Carol,David,Ed,Frank,Ged,Hank,Iona,James,Kelly,Larry,Mark}\}$$

Define a function $f: \mathcal{P}(A) \to \boldsymbol{Z}$ by

Domain is $P(A)$

codomain Z

$$f(S) = \begin{cases} 1 & \text{if } S \text{ has 3 elements} \\ 0 & \text{otherwise} \end{cases}$$

Then

$f(\underbrace{\{\text{Carol,Hank,James,Larry}\}}_{\text{4 names}}) = 0$

$f(\underbrace{\{\text{Carol,Hank,James}\}}_{\text{3 names}}) = 1$

$\cancel{f(S) =}$ $f(A) = 0$

set A has 13 names in it

$f(\phi) = 0$

no elements

Even functions that involve just numbers can be complicated by having the numbers part of more complicated structures. For instance, functions can have input or output consisting of *ordered n-tuples* of numbers.

[Example 4] (similar to 7.1#12) Let $J_{10} = \{0,1,2,3,4,5,6,7,8,9\}$

Define a function $f: J_{10} \times J_{10} \times J_{10} \to J_{10} \times J_{10}$ by

$$f(a, b, c) = \big((a + b + c) mod\ 10, (abc) mod\ 10\big)$$

Then

$f((4,7,2)) = \big((4+7+2) \bmod 10, (4\cdot 7\cdot 2) \bmod 10\big)$

input ordered triple

output ordered pair

$= (13 \bmod 10, 56 \bmod 10)$

$= (3, 6)$

Some common mathematical functions have descriptions that are not simple formulas.

Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically:

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

You have a homework question (7.1#18) involving various logarithm functions and their corresponding exponential functions.

Projections

We started this video with a discussion of the identity function on a set. Although it is a very simple function, it is often very important. Another simple but useful class of functions comes up when dealing with cartesian products of sets. The functions are called projections, and they have to do with basically leaving out some of the coordinates in a cartesian product, to produce a new cartesian product with fewer coordinates. We will only discuss a simple example of projection here.

The Projection onto the $\boldsymbol{k^{th}}$ Coordinate

Symbol: p_k or π_k

Spoken: the *projection onto the k^{th} coordinate*

Usage: There is a cartesian product $A_1 \times A_2 \times \cdots \times A_k \times \cdots \times A_m$ in use

Meaning : the function $p_k: A_1 \times A_2 \times \cdots \times A_k \times \cdots \times A_m \to A_k$ defined by

$$p_k\big((a_1, a_2, \dots, a_k, \dots, a_m)\big) = a_k$$

[Example 5](similar to 7.1#15) Let $X = \boldsymbol{Z}^+$ and $Y = \{a, b, c, d, e\}$ and $Z = \{i, ii, iii\}$

Then p_2 is the function $p_2 \colon X \times Y \times Z \to Y$ defined by $p_2(x, y, z) = y$, etc.

So $p_2\big((13, d, i)\big) =$ d

and $p_3\big((13, d, i)\big) =$ i

Functions Involving Strings and Bit Strings

Ordered n-tuples were introduced in Chapter 1. The definition is copied for reference at the start of this video.

A string is a slight variation on the idea of *ordered n-tuples*.

> **Definition**
>
> Let n be a positive integer. Given a finite set A, a **string of length n over A** is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters. It is often denoted λ and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

In the homework assignment H07.1 for this section, you have a problem (7.1#28) that is about functions whose domain and codomain are sets of bit strings of a certain length. That exercise references an example presented in Section 7.1. The point of the exercise is for you to carefully read the example and understand it. I won't discuss the concepts here, except to say that there is no difficult math involved, just careful reading.

Boolean Functions

Boolean functions are introduced on page 432.

> **Definition**
>
> An **(n-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

The book's discussion of the topic is excellent, and there is no need for me to discuss the topic in this video. I will point out that in the book's discussion, you will read that there are multiple ways of presenting a particular Boolean function:

- a formula
- an arrow diagram
- a table of values

One of your homework exercises (7.1#32) is about finding the output values for a three-place Boolean function, and giving an alternate presentation of the function involving a table.

When are two functions equal?

The concept of function equality sounds like it would not be confusing. But in practice, many students do not really understand what it means.

For instance, are these two functions equal?

$$f(x) = \frac{(x+3)(x-2)}{(x-2)} \quad \text{and} \quad g(x) = x+3$$

Many students will say that these two functions are equal. But they are not equal.

Why aren't they equal? What is the criterion for function equality?

The key is to remember that a *function* is a *relation*, which means that a *function* is a *subset* of a *Cartesian product* that satisfies a certain requirement.

That is, a function $f: A \to B$ is a subset of the Cartesian product $A \times B$ with the property that for every $a \in A$, there is exactly one ordered pair (a, b), where $b \in B$, in the set. The element b in the set (a, b) is denoted $f(a)$. Using this symbol, we would say that for every $a \in A$, the set f contains exactly one ordered pair $\big(a, f(a)\big)$, where $f(a) \in B$.

So a function f is a set of ordered pairs.

And a function g is a set of ordered pairs.

We already know what it means to say that two sets are equal: they contain exactly the same elements. Therefore, to say that two functions f and g are equal means that they contain exatly the same ordered pairs. But that means that they have to have the same domain A and codomain B, and for every $a \in A$, the elements $f(a) = g(a)$.

We see that the functions

$$f(x) = \frac{(x+3)(x-2)}{(x-2)} \quad and\ g(x) = (x+3)$$

are not equal because they do not have the same domain.

The domain of g is the set all real numbers $\mathbb{R}$

The domain of f is the set all real numbers except $x=2$. $\mathbb{R} - \{2\}$

So the functions are not equal.

$$g(2) = 2+3 = 5$$

$$f(2) = \frac{(2+3)(2-2)}{(2-2)} = \frac{5 \cdot 0}{0} = \frac{0}{0}$$ does not exist

You have a homework problem (7.1#14) involving two functions whose formulas involve the floor and ceiling functions, which were introduced in Section 4.6

$\lfloor x \rfloor$

Definition

Given any real number x, the **floor of x**, denoted $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{that unique integer } n \text{ such that } n \le x < n+1.$$

Symbolically, if x is a real number and n is an integer, then

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \le x < n+1.$$

$\lceil x \rceil$

Definition

Given any real number x, the **ceiling of x**, denoted $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \text{that unique integer } n \text{ such that } n-1 < x \le n.$$

Symbolically, if x is a real number and n is an integer, then

$$\lceil x \rceil = n \quad \Leftrightarrow \quad n-1 < x \le n.$$

For example, $\lfloor \pi \rfloor = 3 \qquad \lceil \pi \rceil = 4$

$\lfloor 5 \rfloor = 5 \qquad \lceil 5 \rceil = 5$

Your homework problem involves the question of whether two given functions involving floor and ceiling are equal. I won't discuss a similar example here.

Images and Preimages

The book has the following definition of the *image of a set* and the *inverse image of a set*.

Definition

If $f: X \longrightarrow Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

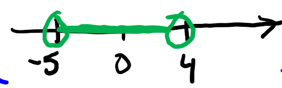
A more common term for *inverse image* is *preimage*.

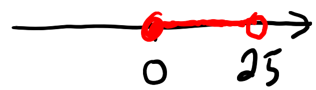
There is some subtlety here, because the notation can be misleading.

[Example 6] For the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = x^2$

(a) Find the image of $\{-5,4\}$ $f(\{-5,4\}) = \{f(-5), f(4)\} = \{25, 16\}$

(b) Find the image of $(-5,4)$

$f((-5,4)) = f($ 

$) =$ 

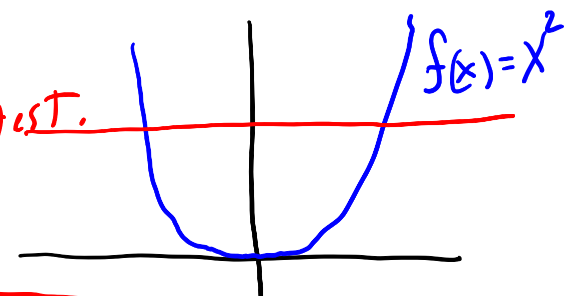
$= [0, 25)$

(c) Find the inverse image of 9 $f^{-1}(9) = \{-3, 3\}$

(d) Find $f^{-1}(0)$ $= \{0\}$

(e) What is the inverse function? $f(x) = x^2$ does not have an inverse function, because $f(x) = x^2$ is not one-to-one.

function fails the horizontal line test.

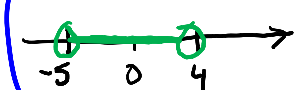


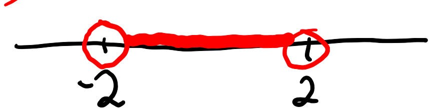
(f) Find the preimage of -5

$f^{-1}(-5) = \emptyset$ no x values have $x^2 = -5$

(g) Find the preimage of $(-5,4)$

this is the correct answer

$f^{-1}($ 

$) =$ 

$= (-2, 2)$

(There is a mistake in the video here.)

Note that the symbol f^{-1} in general does not denote a function in the ordinary sense. It does not take as input a number and produce as output a number.

For instance, the $f(x) = x^2$ does not have an *inverse function* in the ordinary sense. It can't because $f(x) = x^2$ is not one-to-one.

But sometimes the symbol f^{-1} in *does* denote a function in the ordinary sense. For instance, the function $g(x) = x^3$ has an inverse function $g^{-1}(x) = x^{1/3}$

It is interesting to realize that for any function $f: A \to B$, the symbol f^{-1} does actually denote a function of a different sort.

- The domain of f^{-1} is the set of all subsets of B.
- The codomain of f^{-1} is the set of all subsets of A.

That is,

$$f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$$

[Example 7] (similar to 7.2#42)

(a) Prove or Disprove: $\forall F: X \to Y \left(\forall A, B \subseteq X\left(F(A) \cap F(B) \subseteq F(A \cap B)\right)\right)$

False! Let $f(x) = x^2$ So $f: \mathbb{R} \to \mathbb{R}$

Let $A = \{-2\}$ $B = \{2\}$

$f(A) = f(\{-2\}) = \{4\}$

$f(B) = f(\{2\}) = \{4\}$

So $f(A) \cap f(B) = \{4\} \cap \{4\} = \{4\}$

But $A \cap B = \{-2\} \cap \{2\} = \phi$

So $f(A \cap B) = f(\phi) = \phi$

observe $f(A) \cap f(B) \not\subseteq f(A \cap B)$

[Example 7], continued

(b) Prove or Disprove: $\forall F: X \to Y \left(\forall A, B \subseteq X \left(F(A) \cup F(B) \subseteq F(A \cup B)\right)\right)$ (Similar to 7.4 # 42)

True !!

Proof

(1) Suppose $F: X \to Y$ and $A, B \subseteq X$ (generic particular element.)

(2) Suppose $y \in f(A) \cup f(B)$

(3) Then $y \in f(A)$ or $y \in f(B)$ ((2) + definition of union)

Case 1 when $y \in f(A)$

(4) Suppose $y \in f(A)$

(5) There exists an $a \in A$ such that $y = f(a)$ (by 4 and definition of image of a set)

(6) But $a \in A \cup B$ as well by definition of union

(7) But that means $y \in f(A \cup B)$ (by 5, 6, and definition of image of a set.

in this case.

Case 2 when $y \in f(B)$

(8) Suppose $y \in f(B)$

(9) There exists a $b \in B$ such that $y = f(b)$
(by (8) and definition of image of a set.)

(10) But then $b \in A \cup B$ (by (9) and definition of union)

(11) So $y \in f(A \cup B)$ in this case (by (9), (10) and definition of image of a set,

Conclusion of cases

(12) therefore $y \in f(A \cup B)$ (by 3, 7, 11 and proof by division into cases)

(*) Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$

End of Proof