**Video for Homework H07.2 Injective, Surjective, Bijective, and Inverse Functions**

**Reading:** Section 7.2 One-to-One Functions, Onto Functions, Inverse Functions

**Homework:** H07.2: 7.2#5,7,12,17,41,49

**Topics:**

- **Injective Functions**
- **Surjective Functions**
- **Bijective Functions**
- **Inverse Functions**

## Injective Functions

**Definition of Injective Function**

**Words:** $f$ *is* ***injective***, or $f$ *is an* ***injection***, or $f$ is ***one-to-one***

**Usage:** $f$ is a function, $f: X \to Y$

**Meaning:** If two inputs cause outputs that are equal, then the inputs must be equal.

**Meaning Written Formally:** $\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)$

**Contrapositive Meaning:** If two inputs are not equal, then the outputs will not be equal.

**Contrapositive Written Formally:** $\forall x_1, x_2 \in X\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)$

**Other Wording:** For every element in the codomain, there exists ***at most one*** element of the domain that can be used as input to cause that element of the codomain to be output.

**Other Formal Presentation:** $\forall y \in Y\big(\exists \text{ at most one } x \in X(f(x) = y)\big)$

**Incorrect interpretation of the term *one-to-one***

When students are asked what *one-to-one* means, the answer they always give is always this:

*For each $x$, there is one $y$.*

It is important to realize that the sentence above is *not* what it means to be *one-to-one*. In fact, the sentence above is what it means for $f$ to be a *function*. *Every* function has the property that for each $x$, there is one $y$. It has nothing to do with the property of being *one-to-one*.

Because the term *one-to-one* is so often misunderstood, many mathematicians prefer to use the term *injective* instead. In these videos, I will mostly use the term *injective*.

**What it means to *not* be *injective***

It will be important to know how to determine when a function is injective or not injective. For that, it will be necessary to know the negation of the formal statement that $f$ is injective.

$f$ is injective: $\forall x_1, x_2 \in X \left( \text{If } f(x_1) = f(x_2) \text{ then } x_1 = x_2 \right)$

$f$ is not injective:

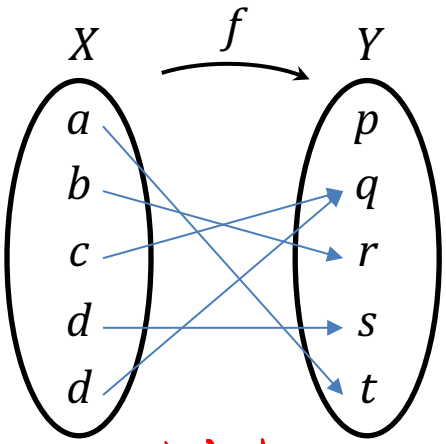
$$\sim \left( \forall x_1, x_2 \in X \left( \text{If } f(x_1) = f(x_2) \text{ then } x_1 = x_2 \right) \right)$$

$$\equiv \exists x_1, x_2 \in X \left( f(x_1) = f(x_2) \text{ and } x_1 \neq x_2 \right)$$

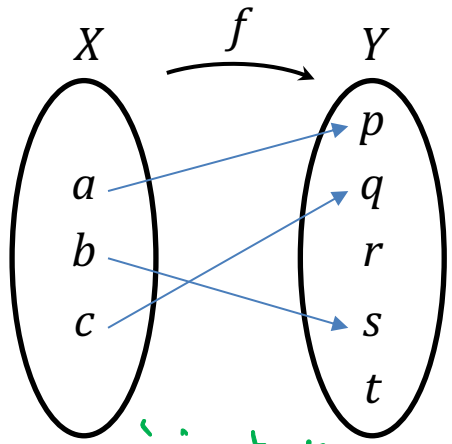
There exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$

and yet they cause the same output $f(x_1) = f(x_2)$

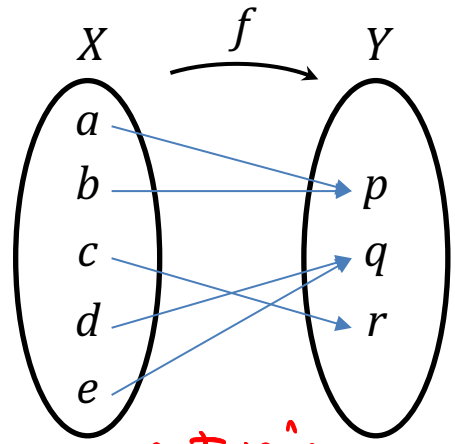
**[Example 1]**



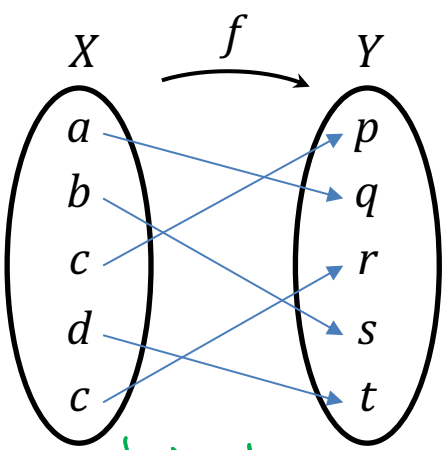

not injective
not surjective
not bijective



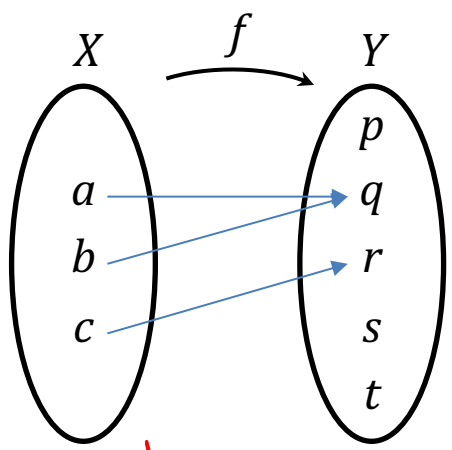

injective
not surjective
not bijective



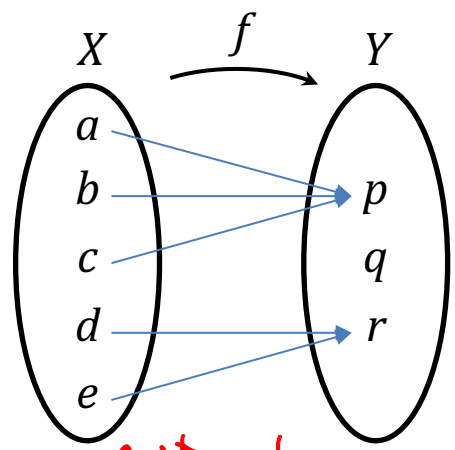

not injective
Surjective
not bijective




injective
Surjective
bijective




not injective
not Surjective
not bijective




not injective
not surjective
not bijective

**Graphs of *functions* and of *injective* and *non-injective* functions.**

A function whose domain and codomain are subsets of the real numbers $\boldsymbol{R}$ can be illustrated using a graph. But there are graphs that do not correspond to functions. The *vertical line test* articulates which graphs qualify to be graphs of functions

**The *Vertical Line Test for a Graph to be the Graph of a Function***

- If, for every $a \in A$, the vertical line $x = a$ intersects the graph exactly once, then the graph is the graph of a function with domain $A$. (The graph *passes* the *vertical line test*.)
- If there exists an $a \in A$ such that the vertical line $x = a$ does not intersect the the graph, or intersects the graph more than once, then the graph is not the graph of a function with domain $A$. (The graph *fails* the *vertical line test*.)

The injective or non-injective property of the function has corresponding behavior in the graph of the function. The correspondence is the essence of what is often called the *horizontal line test*. The name actually needs to be more precise.

**The *Horizontal Line Test for Injectivity***

Suppose $A \subseteq \boldsymbol{R}$ and $B \subseteq \boldsymbol{R}$ and $f: A \to B$

| *injectivity* of $f$ | $\leftrightarrow$ | behavior of the graph of $f$ |
|---|---|---|
| $f$ is *injective* | $\leftrightarrow$ | For every $b \in B$,the horizontal line $y = b$ intersects the graph of $f$ at most once. ($f$ *passes* the *horizontal line test*.) |
| $f$ is *not injective* | $\leftrightarrow$ | There exists a $b \in B$ such that the horizontal line $y = b$ intersects the graph of $f$ more than once. ($f$ *fails* the *test*.) |

**Determining whether a function given by a formula is injective.**

**[Example 2]** Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) = x^2$.

**(a)** Is $f$ *injective*? (Is ~~g~~ f *one-to-one*?) Prove or give a counterexample. Illustrate using a graph.

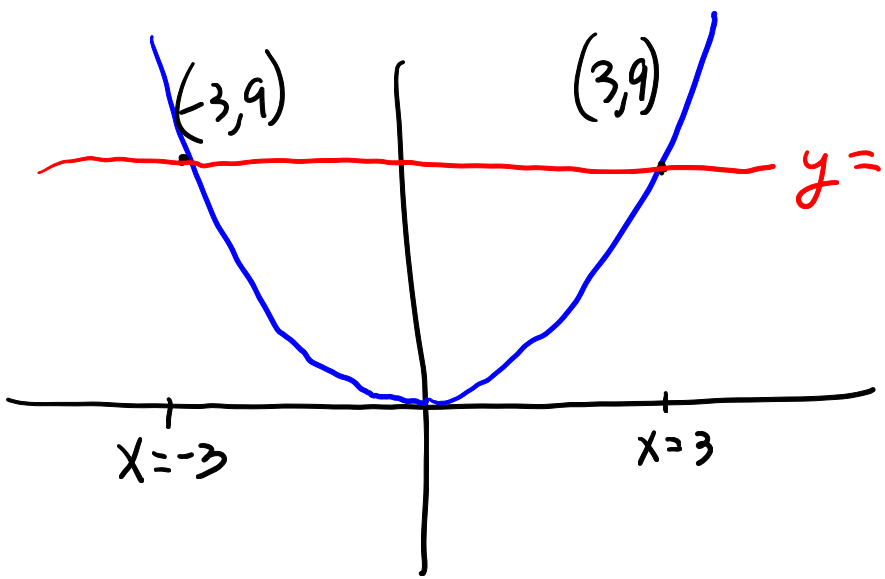
Not one-to-one, Not injective

Let $x_1 = -3$, $x_2 = 3$

Notice $-3 \neq 3$

but $(-3)^2 = (3)^2$

$f(-3) = f(3)$




intersects the graph twice

**[Example 3]** Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) = 4x - 5$.

**(a)** Is $f$ *injective*? (Is $f$ *one-to-one*?) Prove or give a counterexample. Illustrate using a graph.

$f$ is one to one,

Proof must prove $\forall x_1, x_2 \in \mathbb{R}$ (If $f(x_1) = f(x_2)$ then $x_1 = x_2$)

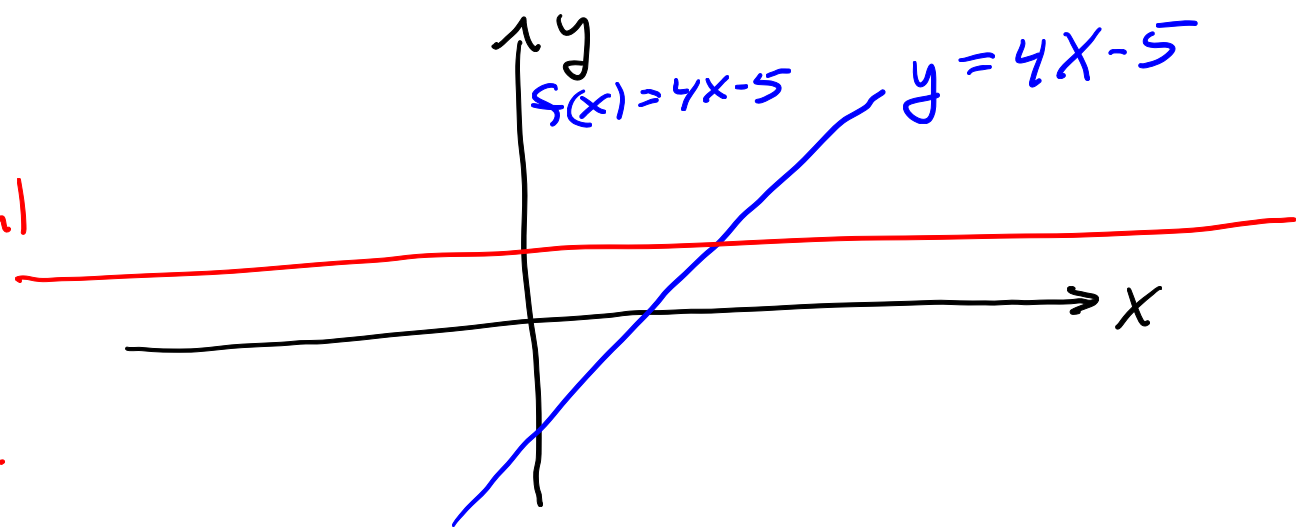
(1) Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$ (generic particular element)

(2) $4x_1 - 5 = 4x_2 - 5$ (by definition of $f(x)$)

(3) $4x_1 = 4x_2$ (added 5 to both sides)

(4) Therefore $x_1 = x_2$ (divided by 4)

End of proof




no horizontal line touches the graph more than once.

**[Example 4]** Let $f(x) = \dfrac{x-2}{x-3}$

**(a)** What is the *domain* of $f(x)$? (the *natural domain*)

all $x \neq 3$ that is $A = \mathbb{R} - \{3\} = \{x \in \mathbb{R} \mid x \neq 3\}$

**(b)** What is the *codomain* of $f(x)$?

The set $\mathbb{R}$

**(c)** What is the range of $f(x)$? That is, what is the following set?

$$range(f) = image\ of\ A = f(A) = \{y = f(x) | x \in A\}$$

To answer that question, it helps to think of the formula for $f(x)$ as an equation involving $x$ and $y$ and and to solve the equation for $x$.

$$y = \frac{x-2}{x-3}$$

multiply by $x-3$

$$(x-3)y = x-2$$

$$xy - 3y = x - 2$$

$$xy - x = 3y - 2$$

$$x(y-1) = 3y-2$$

$$x = \frac{3y-2}{y-1}$$

Range is all $y \neq 1$

The set $B = \mathbb{R} - \{1\} = \{y \in \mathbb{R} \mid y \neq 1\}$

**(d)** Is $f$ *injective*? (Is $f$ *one-to-one*?) Prove or give a counterexample.

yes

must prove $\forall x_1, x_2 \in A$ (If $f(x_1) = f(x_2)$ then $x_1 = x_2$)

Proof

Direct Proof

(1) Suppose $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ (generic particular elements)

(2) Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. So $y_1 = y_2$

So $x_1 = \dfrac{3y_1 - 2}{y_1 - 1}$ and $x_2 = \dfrac{3y_2 - 2}{y_2 - 2}$

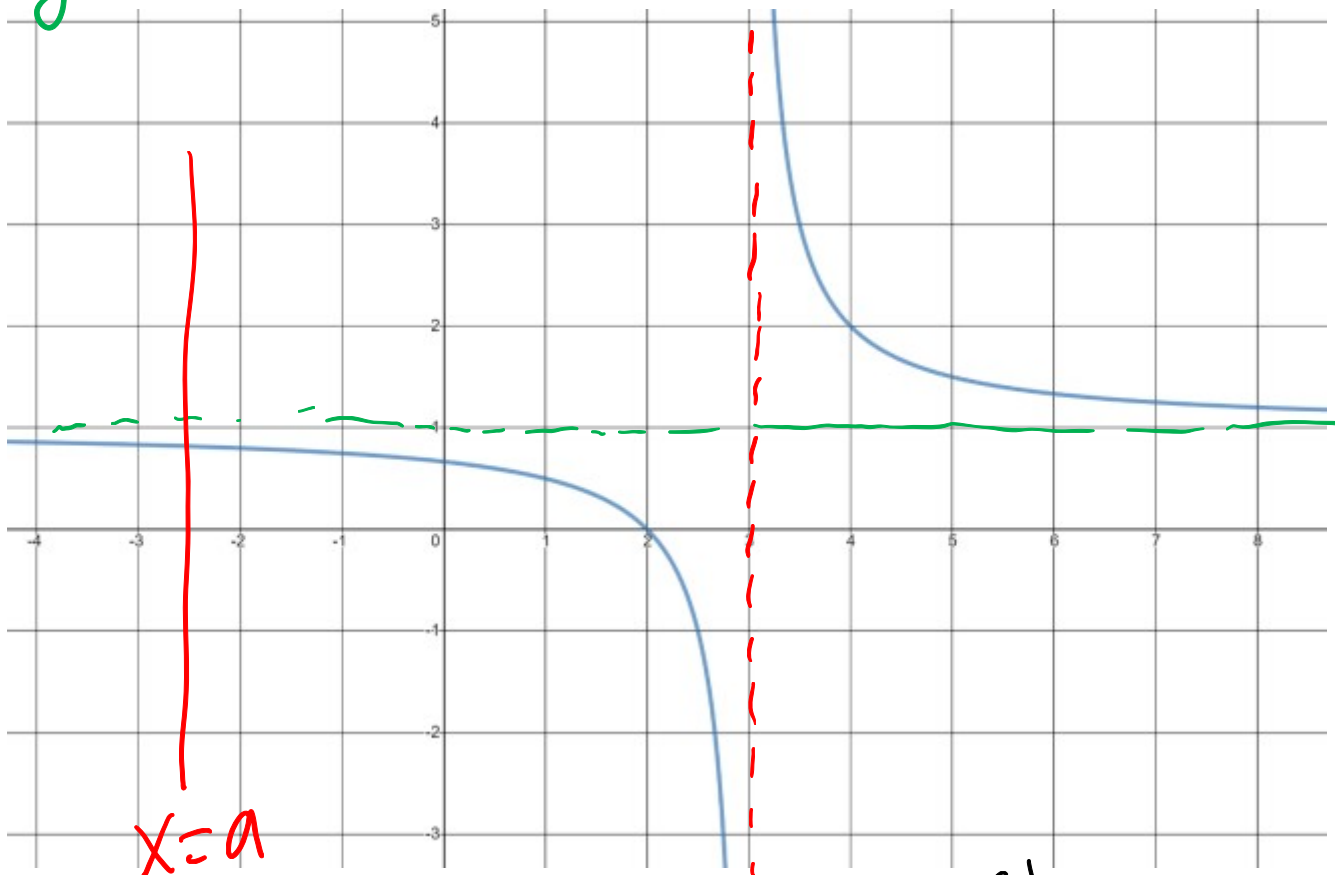
because $y_1 = y_2$, these expressions will yield the same result

(3) therefore $x_1 = x_2$

End of proof

**(e)** Illustrate your results from (a), (c), (d) using this given graph of $f(x)$.

(c) every horizontal line of the form $y=b$ where $b \in B$ does touch the graph




(a) every vertical line of the form $x=a$ where $a \in A$ intersects the graph exactly once

(d) There are <u>no</u> horizontal lines that touch the graph more than once

## Surjective Functions

**Definition of Surjective Function**

**Words:** $f$ *is* ***surjective***, or $f$ *is a* ***surjection***

**Alternate Words:** $f$ *is* ***onto***

**Usage:** $f$ is a function, $f: X \to Y$

**Meaning:** For every element in the codomain, there exists an element of the domain (at least one) that can be used as input to cause that element of the codomain to be output.

**Meaning Written Formally:** $\forall y \in Y\big(\exists x \in X(f(x) = y)\big)$

**What it means to *not* be *surjective***

It will be important to know how to determine when a function is surjective or not surjective. For that, it will be necessary to know the negation of the formal statement that $f$ is surjective.

$f$ is surjective: $\forall y \in Y (\exists x \in X (f(x) = y))$

$f$ is not surjective: $\sim(\forall y \in Y (\exists x \in X (f(x) = y)))$

$\exists y \in Y (\forall x \in X (f(x) \neq y))$

There exists an element $y$ in the codomain $Y$ that <u>cannot</u> be achieved as the output from some input $x \in X$

**[Example 1](continued)** Return to **[Example 1]** and indicate which functions are *surjective*.

**Graphs of *surjective* and *non-surjective* functions.**

Earlier we saw that for a function whose domain and codomain are subsets of the real numbers, there is a *horizontal line test for injectivity*. Observe that it is possible to articulate a similar test for *surjectivity*.

**The *Horizontal Line Test for Surjectivity***

Suppose $A \subseteq \boldsymbol{R}$ and $B \subseteq \boldsymbol{R}$ and $f: A \to B$

| *surjectivity* of $f$ | $\leftrightarrow$ | behavior of the graph of $f$ |
|---|---|---|
| $f$ is *surjective* | $\leftrightarrow$ | For every $b \in B$,the horizontal line $y = b$ intersects the graph of $f$ at least once. ($f$ *passes* the *horizontal line test*.) |
| $f$ is *not surjective* | $\leftrightarrow$ | There exists a $b \in B$ such that the horizontal line $y = b$ does not intersect the graph of $f$. ($f$ *fails* the *test*.) |

Return to the functions presented in earlier examples and indicate whether each is surjective.

**[Example 2](continued)** Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) = x^2$.

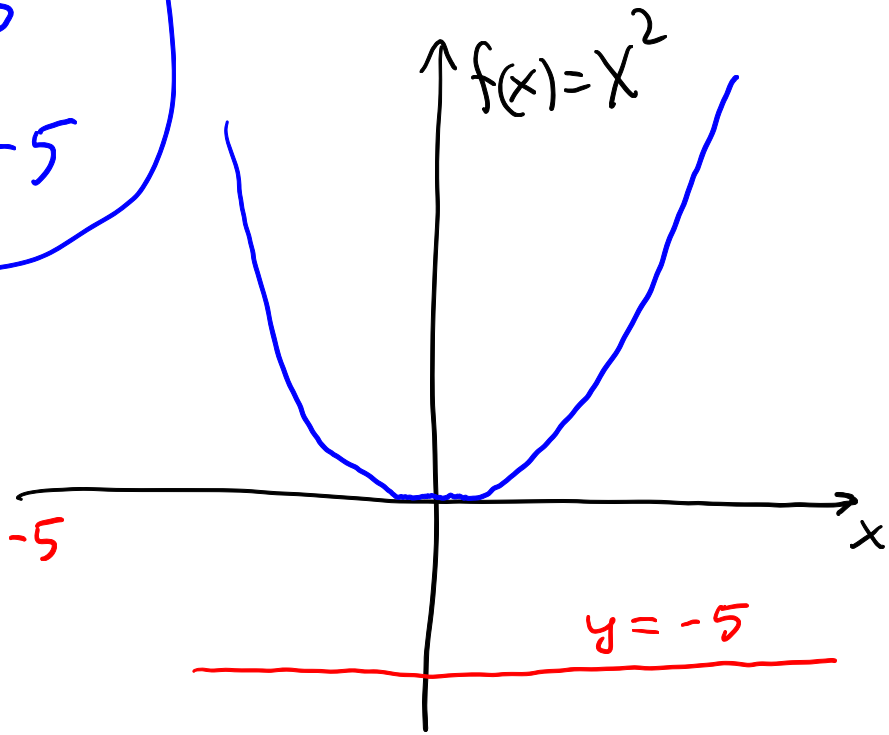
**(b)** Is $f$ *surjective*? (Is $f$ *onto*?) Prove or give a counterexample. Illustrate using a graph.

$f(x)$ is not surjective. (not onto)

Let $y = -5$ (counterexample)

There is no $x$ such that $f(x) = -5$

There is no $x$ such that $x^2 = -5$




The horizontal line $y = -5$ does not intersect the graph

not a bijection, either

**[Example 3](continued)** Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) = 4x - 5$.

**(b)** Is $f$ *surjective*? (Is $f$ *onto*?) Prove or give a counterexample. Illustrate using a graph.

f is surjective

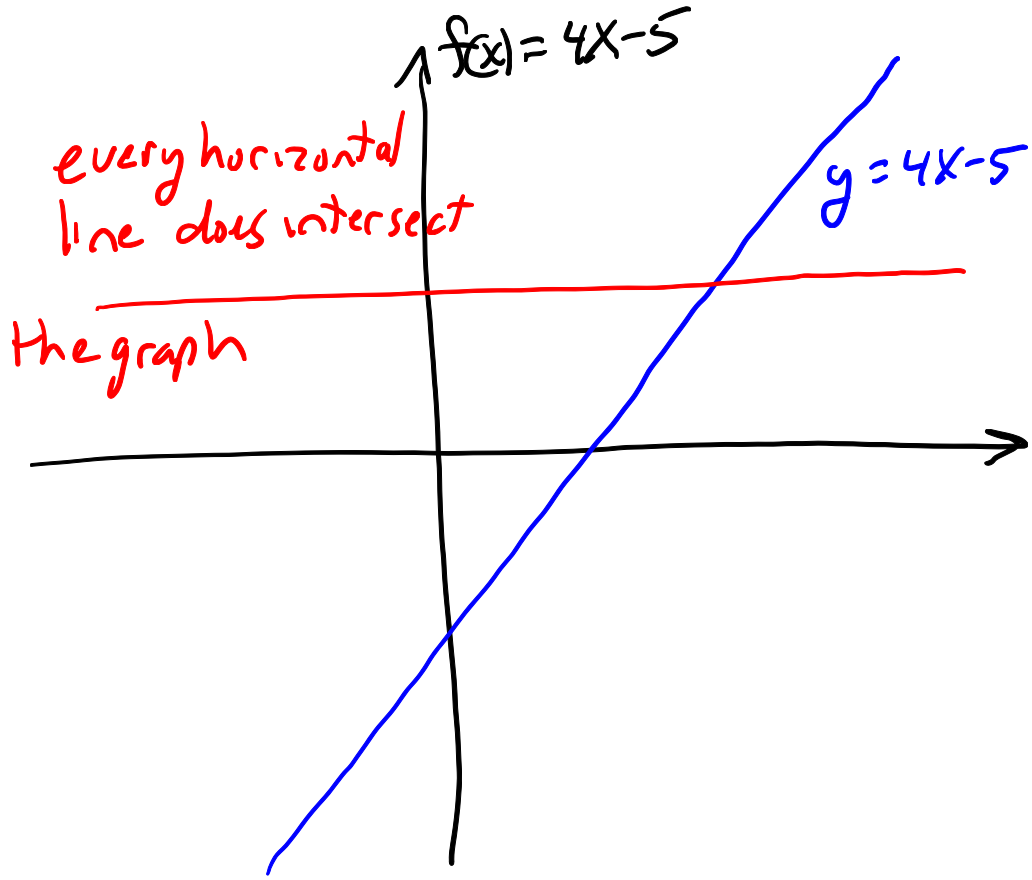
To prove this, consider $f(x) = 4x - 5$ as an equation involving x & y

$$y = 4x - 5$$

Solve for x

$$y + 5 = 4x$$

$$\frac{y+5}{4} = x$$




f is bijective as well

Proof that $f(x)$ is surjective

Proof that $\underbrace{\forall y \in \mathbb{R}}(\underline{\exists x \in \mathbb{R}(f(x) = y)})$

(1) Suppose $y \in \mathbb{R}$ is given (generic particular element)

(2) Let $x = \frac{y+5}{4}$ (result of solving the equation for $x$)

Then $f(x) = f\left(\frac{y+5}{4}\right)$

$$= \cancel{4}\left(\frac{y+5}{\cancel{4}}\right) - 5$$

$$= y + 5 - 5$$

$$= y$$

(3) Therefore $\exists x \in \mathbb{R}(f(x) = y)$

End of proof

**[Example 4](continued)** Let $f(x) = \dfrac{x-2}{x-3}$

**(f)** Is $f$ *surjective*? (Is $f$ *onto*?) Prove or give a counterexample.

The domain of $f(x)$ is set $A = \{x \in \mathbb{R} \mid x \neq 3\}$

The codomain is the set $\mathbb{R}$

The range (the set of actual outputs) is the set

$$B = \{y \in \mathbb{R} \mid y \neq 1\}$$

The number $y=1$ is in the codomain, but not in the range. There is no $x$ value such that $f(x) = 1$. So $f(x)$ is <u>not</u> surjective

**(g)** Illustrate your result from **(f)** using this given graph of $f(x)$.

**A function's properties depend on the choice of domain and codomain.**

**[Example 4](c)** Recall that the function from **[Example 4]**

$$f(x) = \frac{x-2}{x-3}$$

has domain $A = \boldsymbol{R} - \{3\} = \{x \in \boldsymbol{R} | x \neq 3\}$

In function notation, we write $f: A \to \boldsymbol{R}$

Notice that we cannot write $f: \boldsymbol{R} \to \boldsymbol{R}$, because the formula for $f(x)$ does not give a $y$ value when $x = 3$. The domain cannot be the set of all real numbers. We would say that the formula

$$f(x) = \frac{x-2}{x-3}$$

does not give a *well-defined* function on the set of all real numbers. It fails to do what a function is required to do.

The conclusion from this is that the choice of domain is important when describing a function.

Also recall that we found that $f$ was not surjective: For the number $y = 1$ in the codomain, there is no $x$ such that $f(x) = 1$.

We can describe this using the terminology of the range and the codomain.

- The *codomain* of $f$ is the set of all real numbers $\boldsymbol{R}$.
- The *range* of $f$ is the set $\boldsymbol{R} - \{1\} = \{y \in \boldsymbol{R} | y \neq 1\}$

We see that the range is not equal to the codomain. (The range is a *proper subset* of the codomain.)

$$range(f) \subsetneq codomain(f)$$

Realize that if we define a set $B$ as follows

$$B = \boldsymbol{R} - \{1\} = \{y \in \boldsymbol{R} | y \neq 1\}$$

And then use just the set $B$, rather than all of $\boldsymbol{R}$, for the codomain, then the resulting function

$$f: A \to B$$

*is* surjective!

The conclusion from this is that the choice of codomain is important when describing a function.

Now return to the function from **[Example 2]**, $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = x^2$.

We found that the function was neither injective nor surjective.

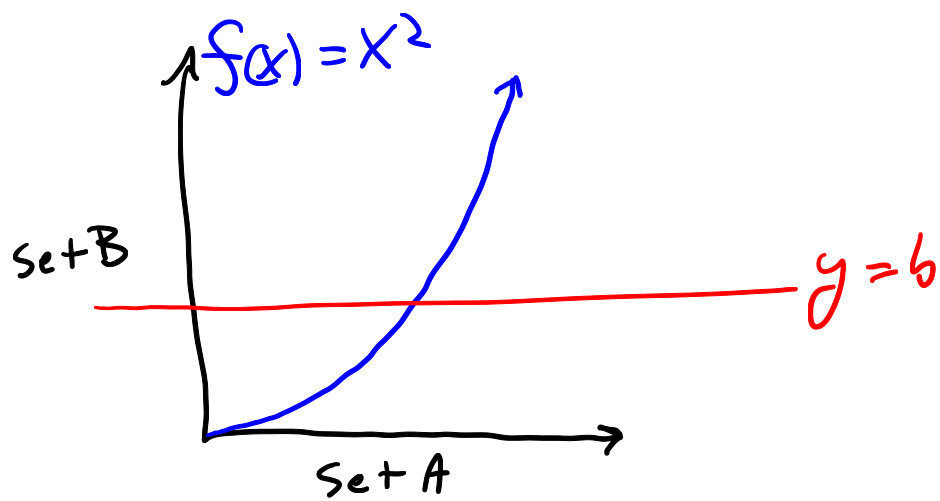
But observe that if we define the set $A$

$$A = \{x \in \mathbb{R} \mid x \geq 0\}$$

and define the set $B$

$$B = \{y \in \mathbb{R} \mid y \geq 0\}$$

Then the function $f: A \to B$ defined by $f(x) = x^2$ is both injective and surjective!



**[Example 3](continued)**

**(c)** Now return to the function from **[Example 3]**, $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = 4x - 5$.

What if we change the domain and codomain to the set $\boldsymbol{Z}$. That is, we have

$$f: \boldsymbol{Z} \to \boldsymbol{Z} \text{ defined by } f(n) = 4n - 5$$

Then f(n) is still injective

Suppose $f(n_1) = f(n_2)$

$4n_1 + 5 = 4n_2 + 5$

$4n_1 = 4n_2$

$n_1 = n_2$

But f(n) is not surjective

Consider the integer $y = 2$

In order to get $f(x) = y = 2$, we would need to use $x = \frac{y+5}{4} = \frac{2+5}{4} = \frac{7}{4}$ But that is not integer!

So there is no integer n such that $f(n) = 2$.

## Bijective Functions

**Definition of Bijective Function**

**Words:** $f$ *is* ***bijective***, or $f$ *is a* ***bijection*** or $f$ *is a* ***one-to-one correspondence***

**Usage:** $f$ is a function, $f: X \to Y$

**Meaning:** $f$ is both *injective* and *surjective*. ($f$ is both *one-to-one* and *onto*.)

### What it means to *not* be *bijective*

It will be important to know how to determine when a function is bijective or not bijective.

For that, it will be necessary to know the negation of the statement that $f$ is bijective.

$f$ is *bijective*: f is injective and f is surjective

$f$ is *not bijective*:

$\sim$ ( f is injective and f is surjective )

$\equiv$ f is not injective or f is not surjective.

**[Examples 1,2,3,4] continued**

Return to the functions presented in **[Examples 1,2,3,4]** and indicate whether each is bijective.

[Example 4] $f(x) = \dfrac{x-2}{x-3}$

Let $A = \{x \in \mathbb{R} \mid x \neq 3\}$

Let $B = \{x \in \mathbb{R} \mid y \neq 1\}$

We found

$f: A \to \mathbb{R}$ is injective but not surjective, so not bijective

But

$f: A \to B$ is both injective and surjective, so bijective

**What would be a *formal presentation* of the meaning of *bijective*?**

To answer that question, we should review the definitions of *surjective* and *injective*.

> ***f* is surjective:** For every element in the codomain, there exists ***at least one*** element of the domain that can be used as input to cause that element of the codomain to be output.
> ***f* is injective:** For every element in the codomain, there exists ***at most one*** element of the domain that can be used as input to cause that element of the codomain to be output.

We can combine these into one concise statement:

> ***f* is *bijective*:** For every element in the codomain, there exists ***exactly one*** element of the domain that can be used as input to cause that element of the codomain to be output.

**Meaning of *Bijective* Written Formally:**

$$\forall y \in Y \left( \exists! x \in X \left( f(x) = y \right) \right)$$

In light of this discussion, it is worthwhile to add some lines to our definition of bijective:

**Definition of Bijective Function (updated with new lines)**

**Words:** $f$ *is* ***bijective***, or $f$ *is a* ***bijection*** or $f$ *is a* ***one-to-one correspondence***

**Usage:** $f$ is a function, $f: X \to Y$

**Meaning:** $f$ is both *injective* and *surjective*. ($f$ is both *one-to-one* and *onto*.)

**Other Wording:** For every element in the codomain, there exists ***exactly one*** element of the domain that can be used as input to cause that element of the codomain to be output.

**Meaning Written Formally:** $\forall y \in Y\big(\exists!\, x \in X(f(x) = y)\big)$

## Inverse Maps and their graphs

**Definition of the Inverse Map**

Given function $f: X \to Y$, we define the ***inverse map*** $f^{-1}: Y \to X$ by saying that

$$f^{-1}(y) = x \text{ means } f(x) = y$$

In terms of the graph, this tells us that

the point $(b, a)$ is on the graph of $f^{-1}$ whenever the point $(a, b)$ is on the graph of $f$

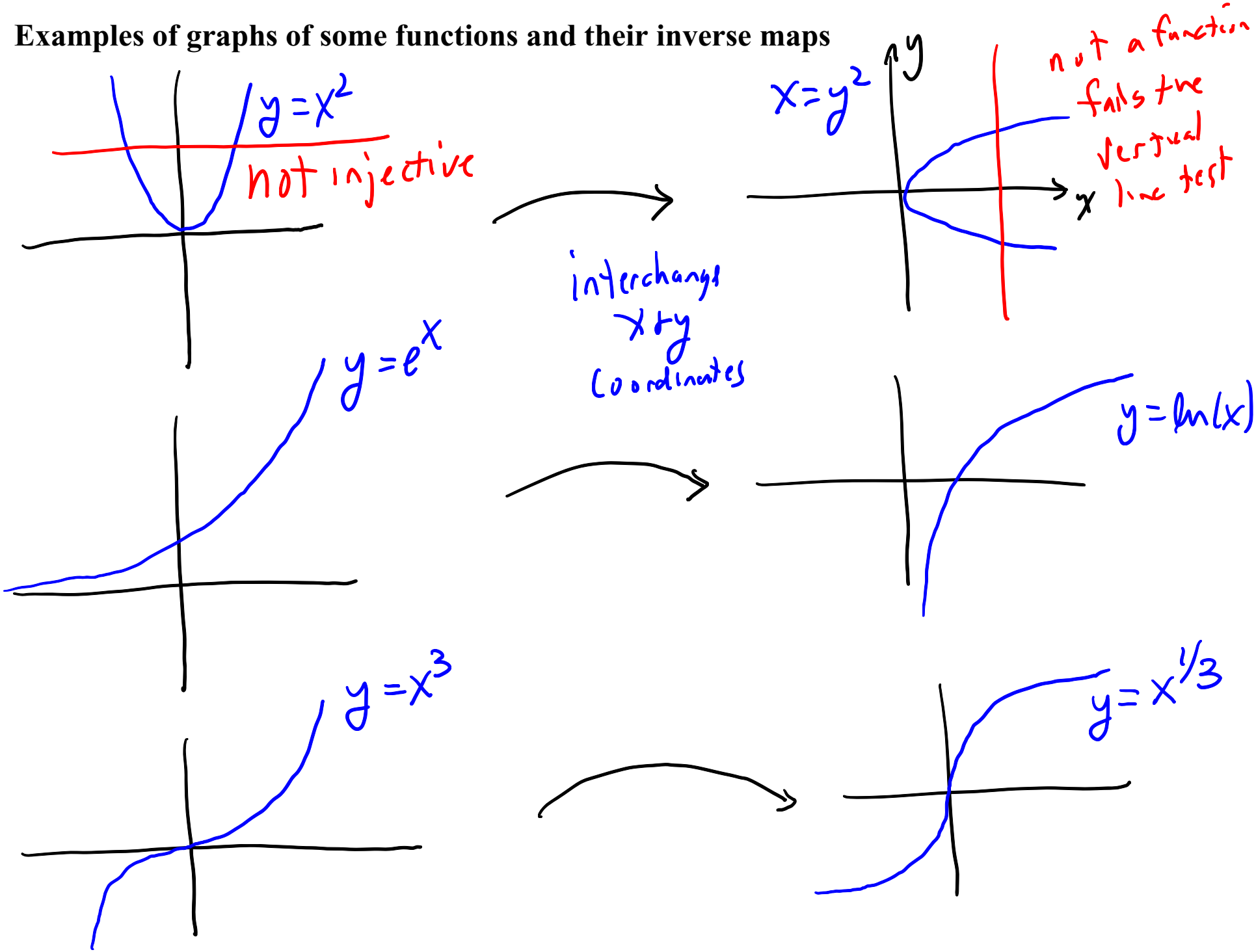
To make a graph of $y = f^{-1}(x)$ by hand

- Start with a graph of $y = f(x)$ and a second, blank $x, y$ axes for the graph of $f^{-1}$
- Find a key point $(x, y) = (a, b)$ on the graph of $f(x)$, and interchange the $x, y$ coordinates to get a new ordered pair $(b, a)$. Plot the ordered pair $(x, y) = (b, a)$ on the axes for $f^{-1}$.
- Repeat this for a bunch of key points.
- When you have enough key points plotted, sketch the graph of $f^{-1}$.

This is equivalent to doing the following:

- Start with a graph of $y = f(x)$ and flip the graph across the line $y = x$ to obtain the graph of $f^{-1}$

**Examples of graphs of some functions and their inverse maps**



**When is an inverse map qualified to be called a function, and what properties will the inverse function have?**

Observe from the three examples above, that given a function $f$.

- The graph of the $f^{-1}$ will pass the vertical line test for function. (that is, $f^{-1}$ will be a function) whenever graph of $f$ passes both horizontal line tests (that is, whenever $f$ is both onto and one-to-one).
- The graph of $f^{-1}$ will automatically pass both horizontal line tests (that is, $f^{-1}$ will automatically be both onto and one-to-one) because the graph of $f$ passes the vertical line test (that is, because $f$ is a function).

**We have convinced ourself of the truth of Theorem 7.2.2 and Theorem 7.2.3**

**Theorem 7.2.2 When an Inverse Map will be a Function**

If a function $f: X \to Y$ is *bijective*, then the inverse map $f^{-1}: Y \to X$ defined by saying

$$f^{-1}(y) = x \text{ means } f(x) = y$$

will have the qualifications to be called a *function*.

**Definition of the Inverse Function**

If a function $f: X \to Y$ is *bijective*, then the inverse map $f^{-1}: Y \to X$ (which is guaranteed to be a function by Theorem 7.2.2) is called the ***inverse function*** for $f$.

**Theorem 7.2.2 The Inverse Function will be Both Injective and Surjective**

If a function $f: X \to Y$ is *bijective*,

then its inverse function $f^{-1}: Y \to X$ will also be *bijective*.

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The book gives a proof of Theorem 7.2.2, but I find that a much simpler proof is one of the coolest things ever. Here is my proof of Theorem 7.2.2

**Proof** Proof that if $f$ is bijective, then $f^{-1}$ is a function and $f^{-1}$ is bijective

Suppose that a function $f: X \to Y$ is *bijective*.

Then we have two quantified statements that are true about $f$:

- Because $f$ is a *function*, we know that $\forall x \in X \left(\exists!\, y \in Y\big(y = f(x)\big)\right)$
- Because $f$ is *bijective*, we know $\forall y \in Y \left(\exists!\, x \in X\big(y = f(x)\big)\right)$

But by definition of inverse map $f^{-1}: Y \to X$,

the expression $y = f(x)$ means same thing as the expression $x = f^{-1}(y)$.

Replacing the expression $y = f(x)$ with the expression $x = f^{-1}(y)$ in the above statements,

- We know $\forall x \in X \left(\exists!\, y \in Y\big(x = f^{-1}(y)\big)\right)$. This tells us that $f^{-1}: Y \to X$ is *bijective*.
- We know $\forall y \in Y \left(\exists!\, x \in X\big(x = f^{-1}(y)\big)\right)$. This tells us that $f^{-1}: Y \to X$ is a *function*.

**End of Proof**

**Finding a formula for the inverse function**

Think of the formula for $f(x)$ as a machine that takes x as input and spits out y as output. That is,

$$y = f(x)$$

This gives you an equation involving $x$ and $y$, an equation that is solved for $y$ in terms of $x$.

Solve the equation for $x$ in terms of $y$.

Think of the new equation as a machine that takes $y$ as input and spits out $x$ as output. That equation describes $f^{-1}$ as a function of $y$. That is,

$$x = f^{-1}(y)$$

The resulting expression for $f^{-1}(y)$ can be rewritten using another variable, if you want.

**Remark on changing the variable names**

Students often tell me that they have learned to find the inverse function by this process

- interchange $x$ and $y$ in the equation
- solve for $y$

Some students prefer this, because it produces a function $f^{-1}(x)$ with variable $x$, not $y$.

I prefer my method, of having

- function $y = f(x)$
- inverse function $x = f^{-1}(y)$

This presentation makes it clear that

- $f: X \to Y$ is a function that takes $x$ as input and poduces the output $y$.
- $f^{-1}: Y \to X$ is a function that takes $y$ as input and poduces the output $x$.

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**[Example 3](continued)**

**(d)** For the function from **[Example 3a]**

$$\text{function } f: \boldsymbol{R} \to \boldsymbol{R} \text{ defined by } f(x) = 4x - 5$$

does $f$ have an inverse function? yes, because f is bijective

If so, find the inverse function.

Start with the equation $y = 4x - 5$

Solve for x: $y + 5 = 4x$

$x = \frac{y+5}{4}$

$x = f^{-1}(y) = \frac{y+5}{4}$

$f^{-1}(t) = \frac{t+5}{4}$

$f^{-1}(x) = \frac{x+5}{4}$

**(d)** For the function from **[Example 3c]**

$$\text{function } f: \boldsymbol{Z} \to \boldsymbol{Z} \text{ defined by } f(n) = 4n - 5$$

does $f$ have an inverse function? no not surjective

If so, find the inverse function.

not bijective.

Does not have an inverse function

**[Example 4](continued)**

**(d)** For the function from **[Example 4c]**

set $A = \boldsymbol{R} - \{3\} = \{x \in \boldsymbol{R} | x \neq 3\}$

set $B = \boldsymbol{R} - \{1\} = \{y \in \boldsymbol{R} | y \neq 1\}$

function $f: A \to B$ defined by $f(x) = \frac{x-2}{x-3}$

does $f$ have an inverse function? yes because f is bijective

If so, find the inverse function.

Start with the equation $y = \frac{x-2}{x-3}$

Solve for x

$y(x-3) = x-2$

$xy - 3y = x - 2$

$xy - x = 3y - 2$

$x(y-1) = 3y - 2$

$x = \frac{3y-2}{y-1}$

So $f^{-1}(y) = \frac{3y-2}{y-1}$