Video for Homework H07.3

Reading: Section 7.3 Composition of Functions

Homework: H07.3: 7.3#4,5,7,9,14

Topics:

- **Definition of Function Composition**
- **Examples of Compositions**
- **Compositions of Injective and Surjective Functions**
- **Compositions and Inverses**

Concepts from Section 7.1 that will be important:

The Identity Function on a Set

The Identity Function on a Set

Symbol: I_X or id_X

Spoken: the *identity function* on X

Usage: X is a set

Meaning : a function $I_X: X \to X$ defined by $I_X(x) = x$

The Floor Function

Definition

Given any real number x, the **floor of x**, denoted $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{that unique integer } n \text{ such that } n \leq x < n+1.$$

Symbolically, if x is a real number and n is an integer, then

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \leq x < n+1.$$

Concept from Section 7.2 that will be important: Injective Functions

Definition of Injective Function

Words: f *is* ***injective***, or f *is an* ***injection***, or f is ***one-to-one***

Usage: f is a function, $f: X \to Y$

Meaning: If two inputs cause outputs that are equal, then the inputs must be equal.

Meaning Written Formally: $\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)$

Contrapositive Meaning: If two inputs are not equal, then the outputs will not be equal.

Contrapositive Written Formally: $\forall x_1, x_2 \in X\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)$

Other Wording: For every element in the codomain, there exists ***at most one*** element of the domain that can be used as input to cause that element of the codomain to be output.

Other Formal Presentation: $\forall y \in Y\big(\exists \text{ at most one } x \in X(f(x) = y)\big)$

Concepts from Section 7.2 that will be important: Surjective Functions

Definition of Surjective Function

Words: f *is **surjective***, or f *is a **surjection***

Alternate Words: f *is **onto***

Usage: f is a function, $f: X \to Y$

Meaning: For every element in the codomain, there exists an element of the domain (at least one) that can be used as input to cause that element of the codomain to be output.

Meaning Written Formally: $\forall y \in Y \big(\exists x \in X (f(x) = y)\big)$

Concepts from Section 7.2 that will be important: Bijective Functions

Definition of Bijective Function (updated with new lines)

Words: f *is* ***bijective***, or f *is a* ***bijection*** or f *is a* ***one-to-one correspondence***

Usage: f is a function, $f: X \to Y$

Meaning: f is both *injective* and *surjective*. (f is both *one-to-one* and *onto*.)

Other Wording: For every element in the codomain, there exists ***exactly one*** element of the domain that can be used as input to cause that element of the codomain to be output.

Meaning Written Formally: $\forall y \in Y\big(\exists!\, x \in X(f(x) = y)\big)$

Concepts from Section 7.2 that will be important: Inverse Functions

Theorem 7.2.2 When an Inverse Map will be a Function

If a function $f: X \to Y$ is *bijective*, then the inverse map $f^{-1}: Y \to X$ defined by saying

$$f^{-1}(y) = x \text{ means } f(x) = y$$

will have the qualifications to be called a *function*.

Definition of the Inverse Function

If a function $f: X \to Y$ is *bijective*, then the inverse map $f^{-1}: Y \to X$ (which is guaranteed to be a function by Theorem 7.2.2) is called the ***inverse function*** for f.

Theorem 7.2.2 The Inverse Function will be Both Injective and Surjective

If a function $f: X \to Y$ is *bijective*,

then its inverse function $f^{-1}: Y \to X$ will also be *bijective*.

.

Definition

There is a typo in this Definition. It should say

$f: X \to Y'$ and $g: Y \to Z$ and it should also say that $Y' \subseteq Y$. (See the picture.)

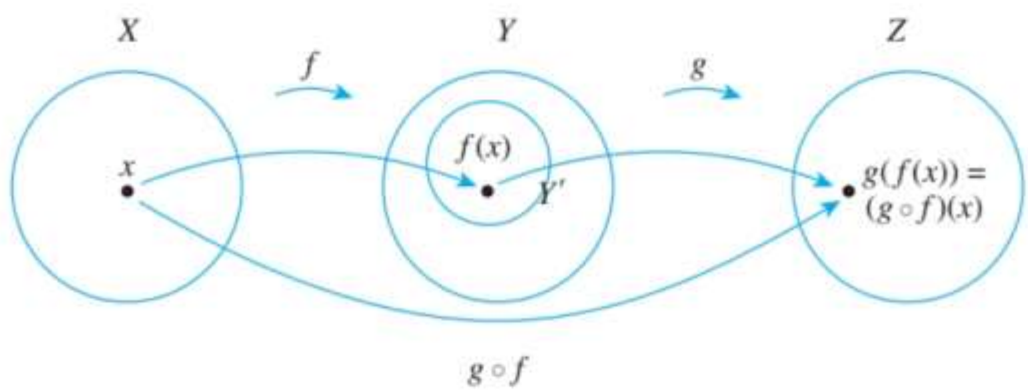
Definition

Let $f: X \to Y$ and $g: Y' \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for each } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g**.

This definition is shown schematically below.



[Example 1] (Similar to 7.3#4)

Define $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = x^3$

Define $g: \boldsymbol{R} \to \boldsymbol{R}$ by $g(x) = x^{1/3}$

(a) Find $f \circ g$

(b) Find $g \circ f$

(c) Does $f \circ g$ equal $g \circ f$?

$f(\ \ ) = (\ \ )^3$ empty version

$g(\ \ ) = (\ \ )^{1/3}$ empty version

ⓐ $f \circ g : \mathbb{R} \to \mathbb{R}$ defined by $(f \circ g)(x) = f(g(x)) = (x^{1/3})^3 = x^{\frac{1}{3} \cdot 3} = x^1 = x$

rule of exponents $(a^b)^c = a^{b \cdot c}$

so $(f \circ g)(x) = x$

ⓑ $g \circ f : \mathbb{R} \to \mathbb{R}$ defined by $(g \circ f)(x) = g(f(x)) = (x^3)^{\frac{1}{3}} = x^{3 \cdot \frac{1}{3}} = x^1 = x$

so $(g \circ f)(x) = x$

ⓒ Observe $f \circ g$ equals $g \circ f$

This illustrates the following theorem.

Theorem 7.3.2 Composition of a Function with Its Inverse

If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then

$$\text{(a) } f^{-1} \circ f = I_X \quad \text{and} \quad \text{(b) } f \circ f^{-1} = I_Y.$$

[Example 2] (Similar to 7.3#4)

Define $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = x^2$

Define $g: \boldsymbol{R}^{nonneg} \to \boldsymbol{R}$ by $g(x) = \sqrt{x} = x^{1/2}$

(a) Find $f \circ g$

(b) Find $g \circ f$

(c) Does $f \circ g$ equal $g \circ f$?

$f(\ \ ) = (\ \ )^2$ empty version

(a) $f \circ g: \mathbb{R}^{nonneg} \to \mathbb{R}$ defined by

$$(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 = (x^{\frac{1}{2}})^2 = x^{\frac{1}{2} \cdot 2} = x^1 = x$$

result: $(f \circ g)(x) = x$

(b) $g \circ f$ is not defined because codomain of f is not a subset of domain of g

$f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R}^{nonneg} \to \mathbb{R}$

$$\mathbb{R} \not\subseteq \mathbb{R}^{nonneg}$$

[Example 2](revised)

Define $f: \boldsymbol{R} \to \boldsymbol{R}^{nonneg}$ by $f(x) = x^2$

Define $g: \boldsymbol{R}^{nonneg} \to \boldsymbol{R}$ by $g(x) = \sqrt{x} = x^{1/2}$

$g(\quad) = (\quad)^{1/2} = \sqrt{(\quad)}$ empty version

(a) Find $f \circ g$

(b) Find $g \circ f$

(c) Does $f \circ g$ equal $g \circ f$?

(a) $f \circ g : \mathbb{R}^{nonneg} \to \mathbb{R}^{nonneg}$ defined by

$(f \circ g)(x) = (\sqrt{x})^2 = x$

Observe $f \circ g$ is equal to $I_{\mathbb{R}^{nonneg}}$ the identity function

(b) $g \circ f : \mathbb{R} \to \mathbb{R}$ defined by

$(g \circ f)(x) = g(f(x)) = (x^2)^{1/2} = |x|$

example $(g \circ f)(5) = \sqrt{5^2} = \sqrt{25} = 5$

but $(g \circ f)(-5) = \sqrt{(-25)^2} = \sqrt{25} = 5$

Observe $(g \circ f)$ is not the identity function!

(c) $f \circ g$ and $g \circ f$ are not equal.
Their domains don't match.

Specific examples

$$(f \circ g)(5) = f(g(5)) = (\sqrt{5})^2 = 5$$

$$(g \circ f)(5) = g(f(5)) = \sqrt{(5)^2} = \sqrt{25} = 5$$

But

$$(f \circ g)(-5) = f(g(-5)) = (\sqrt{-5})^2 \text{ Does not exist}$$

$$(g \circ f)(-5) = g(f(-5)) = \sqrt{(-5)^2} = 5$$

[Example 3](similar to 7.3#7)

Define $f: \boldsymbol{Z} \to \boldsymbol{Z}$ by $f(n) = n^3$

Define $g: \boldsymbol{Z} \to \boldsymbol{Z}$ by $g(n) = n \bmod 4$

(a) Find $f \circ g(3)$ and $g \circ f(3)$

(b) Find $f \circ g(2)$ and $g \circ f(2)$

(c) Does $f \circ g$ equal $g \circ f$?

$f(\ ) = (\ )^3$

$g(\ ) = (\ ) \bmod 4$

empty versions

(a) $f \circ g(3) = f(g(3)) = ((3) \bmod 4)^3 = (3)^3 = 27$

$g \circ f(3) = g(f(3)) = ((3)^3) \bmod 4 = (27) \bmod 4 = 3$

(b) $(f \circ g)(2) = f(g(2)) = ((2) \bmod 4)^3 = (2)^3 = 8$

$(g \circ f)(2) = g(f(2)) = ((2)^3) \bmod 4 = (8) \bmod 4 = 0$

(c) observe $f \circ g$ and $g \circ f$ are not equal!

[Example 4](similar to 7.3#9)

Define $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(n) = 3n$

Define $g: \boldsymbol{R} \to \boldsymbol{R}$ by $g(n) = \lfloor n/3 \rfloor$

(a) Find $f \circ g(3)$ and $g \circ f(3)$

(b) Find $f \circ g(2)$ and $g \circ f(2)$

(c) Does $f \circ g$ equal $g \circ f$?

$f(x) = 3x \qquad f(\ ) = 3(\ )$

$g(x) = \lfloor \frac{x}{3} \rfloor \qquad g(\ ) = \lfloor \frac{(\ )}{3} \rfloor$

empty versions

(a) $f \circ g(3) = f(g(3)) = 3\left(\lfloor \frac{(3)}{3} \rfloor\right) = 3(\lfloor 1 \rfloor) = 3(1) = 3$

$g \circ f(3) = g(f(3)) = \lfloor \frac{3(3)}{3} \rfloor = \lfloor \frac{9}{3} \rfloor = \lfloor 3 \rfloor = 3$

(b) $f \circ g(2) = f(g(2)) = 3\left(\lfloor \frac{(2)}{3} \rfloor\right) = 3\left(\lfloor \frac{2}{3} \rfloor\right) = 3(0) = 0$

$g \circ f(2) = g(f(2)) = \lfloor \frac{3(2)}{3} \rfloor = \lfloor \frac{6}{3} \rfloor = \lfloor 2 \rfloor = 2$

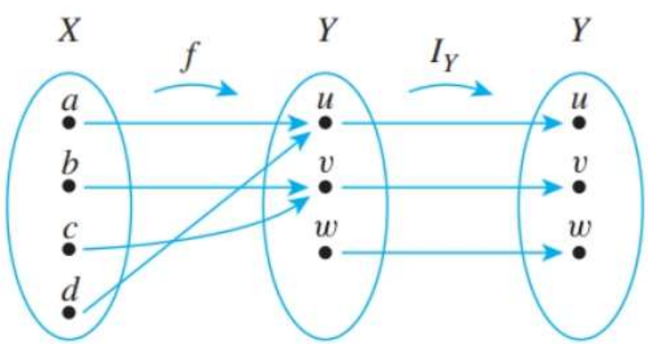
(c) observe $f \circ g$ and $g \circ f$ are <u>not equal</u>.

Not all compositions are difficult:

Theorem 7.3.1 Composition with an Identity Function

If f is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then

$$\text{(a) } f \circ I_X = f \quad \text{and} \quad \text{(b) } I_Y \circ f = f.$$



[Example 5](similar to 7.3#9)

Define sets $A = \{x \in \boldsymbol{R} | x \neq 3\}$ and $B = \{x \in \boldsymbol{R} | x \neq 1\}$

Define $f: A \to B$ by $f(x) = \dfrac{x-2}{x-3}$ $\quad f(\ ) = \dfrac{(\ )-2}{(\ )-3}$

Define g: $B \to A$ by $g(x) = \dfrac{3x-2}{x-1}$ $\quad g(\ ) = \dfrac{3(\ )-2}{(\ )-1}$

(a) Find $f \circ g$

(b) Find $g \circ f$

(a) $f \circ g : B \to B$ is the function

$$(f\circ g)(x) = f(g(x)) = \frac{\left(\frac{3x-2}{x-1}\right) - 2}{\left(\frac{3x-2}{x-1}\right) - 3} = \frac{\left(\left(\frac{3x-2}{x-1}\right) - 2\right)(x-1)}{\left(\left(\frac{3x-2}{x-1}\right) - 3\right)(x-1)} =$$

$$= \frac{(3x-2) - 2(x-1)}{(3x-2) - 3(x-1)} = \frac{3x-2-2x+2}{3x-2-3x+3} = \frac{x}{1} = x$$

Observe $f \circ g$ is I_B

(b) $g \circ f : A \longrightarrow A$ is the function

$$(g\circ f)(x) = \frac{3\left(\frac{x-2}{x-3}\right)-2}{\left(\frac{x-2}{x-3}\right)-1} = \frac{\left(3\left(\frac{x-2}{x-3}\right)-2\right)}{\left(\left(\frac{x-2}{x-3}\right)-1\right)} \cdot \frac{(x-3)}{(x-3)}$$

$$= \frac{3(x-2)-2(x-3)}{(x-2)-1(x-3)} = \frac{3x-6-2x+6}{x-2-x+3} = \frac{x}{1}$$

$$= x$$

Observe: $g \circ f$ is equal to I_A

Remember $f \circ g$ is equal to I_B

Missing Theorem

Theorem 7.3.2 says that the composition of a function with its inverse equals the identity function:

Theorem 7.3.2 Composition of a Function with Its Inverse

If $f: X \rightarrow Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \rightarrow X$, then

$$\text{(a) } f^{-1} \circ f = I_X \quad \text{and} \quad \text{(b) } f \circ f^{-1} = I_Y.$$

There is another theorem that is sort of a converse to Theorem 7.3.2, but it is not stated as a theorem in our book. Rather, it shows up as Exercise 7.3#28. I will state it as a Theorem.

Missing Theorem A (discussed in 7.3 # 28) (sort of the converse of Theorem 7.3.2)

Suppose $f: A \rightarrow B$ and $g: B \rightarrow A$

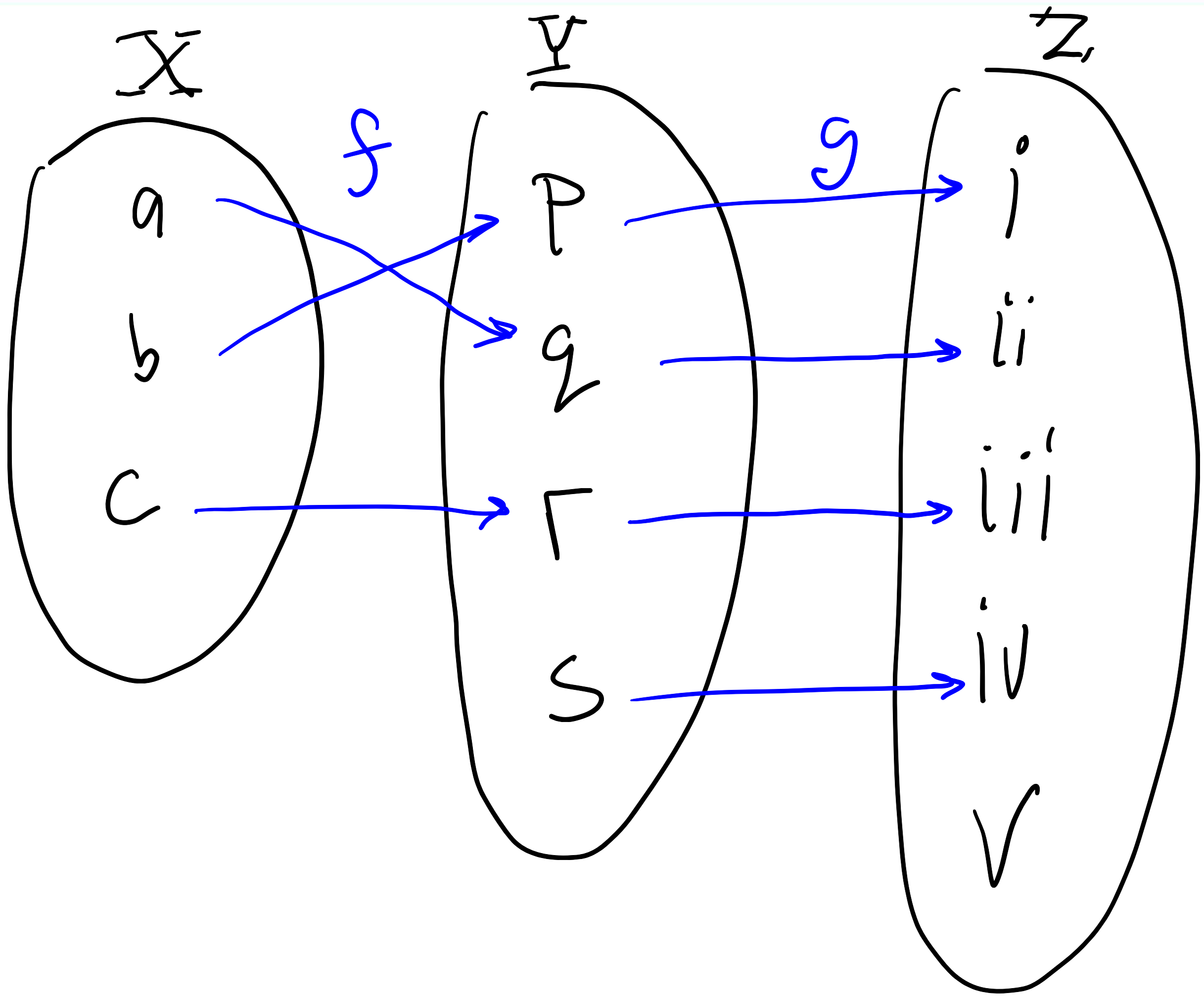
If $g \circ f = id_A$ and $f \circ g = id_B$. then functions f and g are inverses.

Recall that in the video for homework H07.2, we found that the function $f: A \rightarrow B$ defined by $y = f(x) = \frac{x-2}{x-3}$ has inverse function $f^{-1}: B \rightarrow A$ defined by $x = f^{-1}(y) = \frac{3y-2}{y-1}$

Compositions of Injective Functions

Theorem 7.3.3

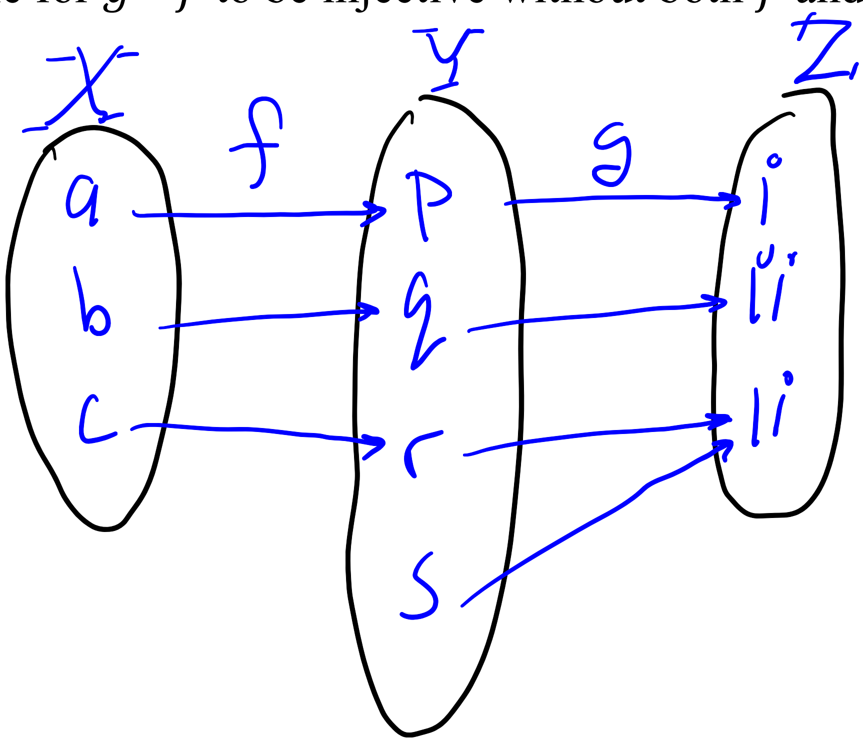
If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one functions, then $g \circ f$ is one-to-one.



The converse statement is not a theorem

Suppose $f: X \to Y$ and $g: Y \to Z$

It is possible for $g \circ f$ to be injective without both f and g being injective



But there is a *partial converse* that is a theorem

There is another theorem that is a *partial converse* to Theorem 7.3.3, but it is not stated as a theorem in our book. Rather, it shows up as Exercise 7.3#21. I will state it as a Theorem.

> **Missing Theorem B (discussed in 7.3 # 21) (*partial converse* of Theorem 7.3.3)**
>
> Suppose $f: X \to Y$ and $g: Y \to Z$
>
> If $g \circ f$ is injective, then f is injective.

Contrapositive: If f is not injective then $g \circ f$ is not injective.

Proof of the contrapositive

(1) Suppose f is not injective

(2) Then there exist x_1, x_2 such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$
(definition of not injective)

(3) But then $g(f(x_1)) = g(f(x_2))$ (because g is a function)

(4) so $(g \circ f)(x_1) = (g \circ f)(x_2)$

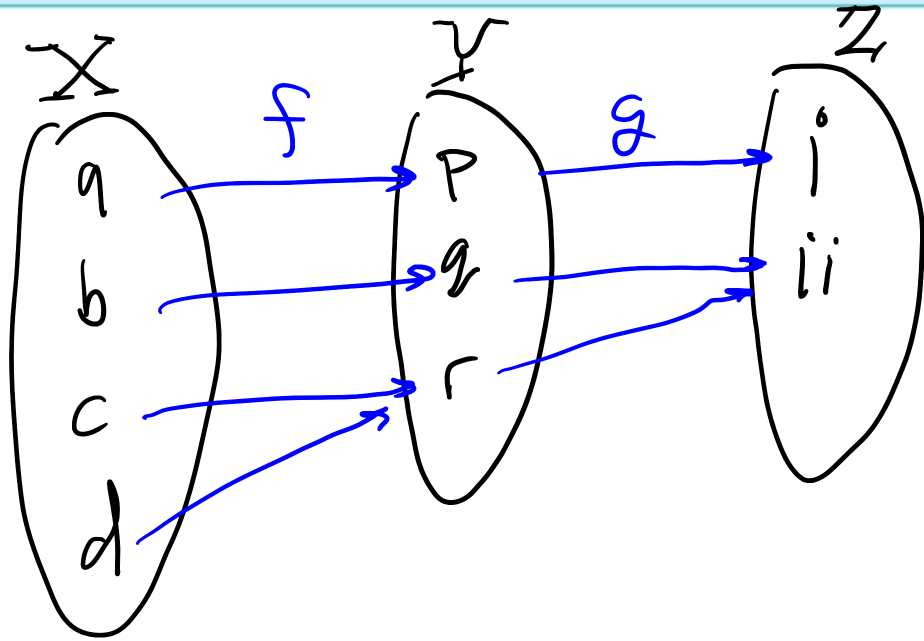
(5) Therefore $g \circ f$ is not injective

End

Compositions of Surjective Functions

Theorem 7.3.4

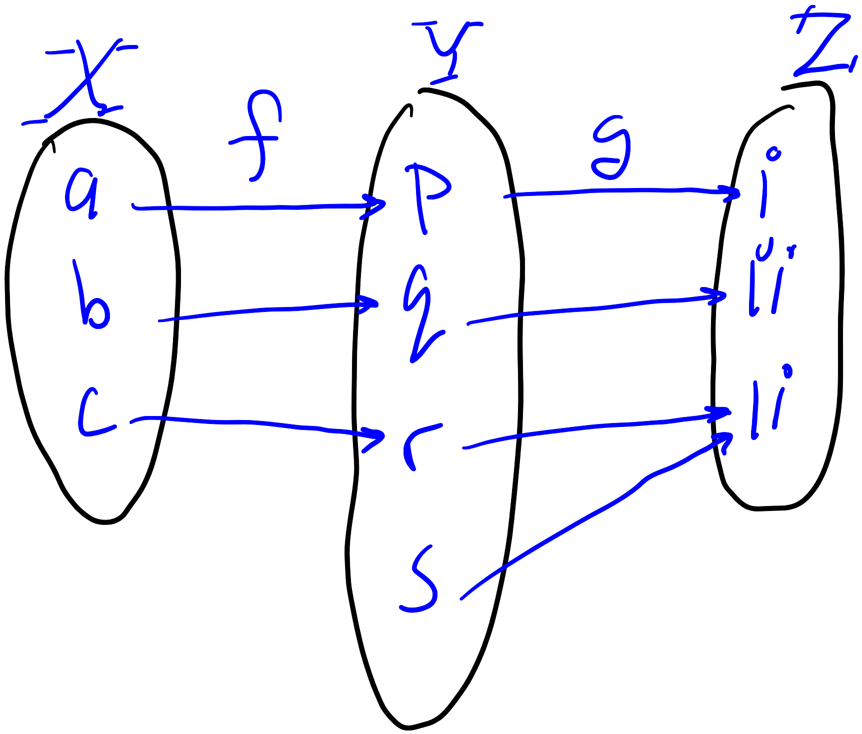
If $f: X \to Y$ and $g: Y \to Z$ are both onto functions, then $g \circ f$ is onto.



The converse statement is not a theorem

Suppose $f: X \to Y$ and $g: Y \to Z$

It is possible for $g \circ f$ to be surjective without both f and g being surjective



But there is a *partial converse* that is a theorem

There is another theorem that is a *partial converse* to Theorem 7.3.4, but it is not stated as a theorem in our book. Rather, it shows up as Exercise 7.3#22. I will state it as a Theorem.

> **Missing Theorem C (discussed in 7.3 # 22) (*partial converse* of Theorem 7.3.4)**
>
> Suppose $f: X \to Y$ and $g: Y \to Z$
>
> If $g \circ f$ is surjective, then g is surjective.

Contrapositive: If g is not surjective then $g \circ f$ is not surjective.

Proof of Contrapositive

(1) Suppose g is not surjective

(2) There exists some $z \in Z$ that cannot be achieved as an output from g. That is, there is no $y \in Y$ such that $g(y) = z$

(3) But then there can be no x in X such that $g(f(x)) = z$.

(4) So there is no x in X such that $(g \circ f)(x) = z$.

(5) Therefore $g \circ f$ is not surjective.

End

[Example 6](7.3#29) A surprising Result about Inverses of Compositions

Suppose $f: X \to Y$ and $g: Y \to Z$ are both bijective

(a) Prove that $g \circ f$ is bijective

(b) Prove that $(g \circ f)^{-1}$ is a function.

(c) Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

ⓐ $g \circ f$ is injective because both g, f are injective (by Theorem 7.3.3)

$g \circ f$ is surjective because both g, f are surjective. (by Theorem 7.3.4)

So $g \circ f$ is bijective.

ⓑ Since $g \circ f$ is bijective, it has an inverse function, (by theorem 7.2.2) denoted $(g \circ f)^{-1}$

(c) Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Part 1

Observe

$$f^{-1} \circ g^{-1} \circ (g \circ f) \overset{\text{associativity}}{=} f^{-1} \circ (g^{-1} \circ g) \circ f$$

$$= f^{-1} \circ I_Y \circ f$$

$$= f^{-1} \circ f$$

$$= I_X$$

Part 2

Observe $\underbrace{(g\circ f)}\circ\underbrace{f^{-1}\circ g^{-1}}$

$$= g\circ(f\circ f^{-1})\circ g^{-1}$$

$$= g\circ I_Y\circ g^{-1}$$

$$= g\circ g^{-1}$$

$$= I_X$$

Conclusion

By Missing theorem A (from exercise 7.3 #28) we can say that $f^{-1}\circ g^{-1}$ and $g\circ f$ are <u>inverses</u>

That is, $(g\circ f)^{-1}$ is $f^{-1}\circ g^{-1}$