Video for Homework H09.1 Introduction to Counting and Probability

Reading: Section 9.1 Introduction to Counting and Probability

Homework: H09.1: 9.1#2,5,10,14,17,22,25,28,32,37,40

Topics:

- **Random Process, Sample Space, Event**
- **Equally Likely Probability Formula**
- **Describing Sample Spaces using Set Notation and other Illustrations**
- **Basic Examples**
- **Choosing with replacement versus choosing without replacement**
- **Counting the Elements of a List**

Random Process, Sample Space, Event, Equally Likely Probability Formula

Random Processes

To say that a process is ***random*** means that when it takes place, one outcome from some set of possible outcomes is sure to occur, and all of the possible outcomes in the set are equally likely to occur.

The ***sample space S*** of a random process is the set of all possible outcomes.

An ***event E*** is a subset of a sample space.

If a sample space S is a finite set, then every event E will also be a finite set. In this case, the symbols $\boldsymbol{N(S)}$ and $\boldsymbol{N(E)}$ denote the ***number of elements*** in S and E. And in this case, the probability of E, denoted $P(E)$ is given by the ***equally likely probability formula***:

$$P(E) = \frac{N(E)}{N(S)}$$

Describing Sample Spaces and Events using Set Notation

[Example 1] (similar to 9.1#11,12,13,14) Flipping a coin four times

A coin is flipped four times and the value of each flip (H or T for Heads or Tails) is recorded.

(a) Describe the sample space S, using set notation. Then find $N(S)$.

$S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$

$N(S) = 16$

(b) Define E to be the event that the number of heads is exactly 2.

Describe event E using set notation. Then, find $N(E)$ and $P(E)$.

$E = \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$

$N(E) = 6$

$P(E) = \frac{N(E)}{N(S)} = \frac{6}{16} = \frac{3}{8} = 0.375$

Illustrating Sample Spaces and Events with Tables or Pictures

Continuing **[Example 1]**

(c) Illustrate the sample space *S* using a table.

HHHH
HHHT
HHTH
HHTT
HTHH
HTHT
HTTH
HTTT
THHH
THHT
THTH
THTT
TTHH
TTHT
TTTH
TTTT

(d) Illustrate the event E on the illustration of the sample space S.

HHHH

HHHT

HHTH

HHTT

HTHH

HTHT

HTTH

HTTT

THHH

THHT

THTH

THTT

TTHH

TTHT

TTTH

TTTT

(e) Define F to be the event that the number of heads is at most 2.

Illustrate the event F on the illustration of the sample space S. Then, find $N(F)$ and $P(F)$.

$N(F) = 11$

$$P(F) = \frac{N(F)}{N(E)} = \frac{11}{16}$$

HHHH

HHHT

HHTH

HHTT *

HTHH

HTHT *

HTTH *

HTTT *

THHH

THHT *

THTH *

THTT *

TTHH *

TTHT *

TTTH *

TTTT *

(f) When flipping a coin four times, what is the most likely number of heads

Explain using your illustration of the sample space.

$P(0) = \frac{1}{16}$

$P(1) = \frac{3}{16}$

$P(2) = \frac{6}{16} = \frac{3}{8}$ most likely

$P(3) = \frac{3}{16}$

$P(4) = \frac{1}{16}$

HHHH	4
HHHT	3
HHTH	3
HHTT	2
HTHH	3
HTHT	2
HTTH	2
HTTT	1
THHH	3
THHT	2
THTH	2
THTT	1
TTHH	2
TTHT	1
TTTH	1
TTTT	0

Drawing a card from a standard deck

A card is drawn from an ordinary deck of 52 cards and the suit and denomination are recorded. Listing the sample space S using set notation is incredibly tedious. So tedious, that one would not want to write out the sample space by hand. And the listing would not be particularly useful, because nobody is going to want to carefully read the set.

{2♠,3♠,4♠,5♠,6♠,7♠,8♠,9♠,10♠,J♠,Q♠,K♠,A♠,
2♣,3♣,4♣,5♣,6♣,7♣,8♣,9♣,10♣,J♣,Q♣,K♣,A♣,
2♥,3♥,4♥,5♥,6♥,7♥,8♥,9♥,10♥,J♥,Q♥,K♥,A♥,
2♦,3♦,4♦,5♦,6♦,7♦,8♦,9♦,10♦,J♦,Q♦,K♦,A♦}

The same goes for a table showing all of the elements of the sample space S explicitly

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

.

But it is possible to illustrate the sample space using a hand-drawn table that does not take so long to draw and that is also very useful.

[Example 2] (similar to 9.1#4,5,6) A card is drawn from an ordinary deck of 52 cards and the suit and denomination are recorded.

(a) Illustrate the sample space S using a table with row and column headings, but empty cells. Find $N(S)$.

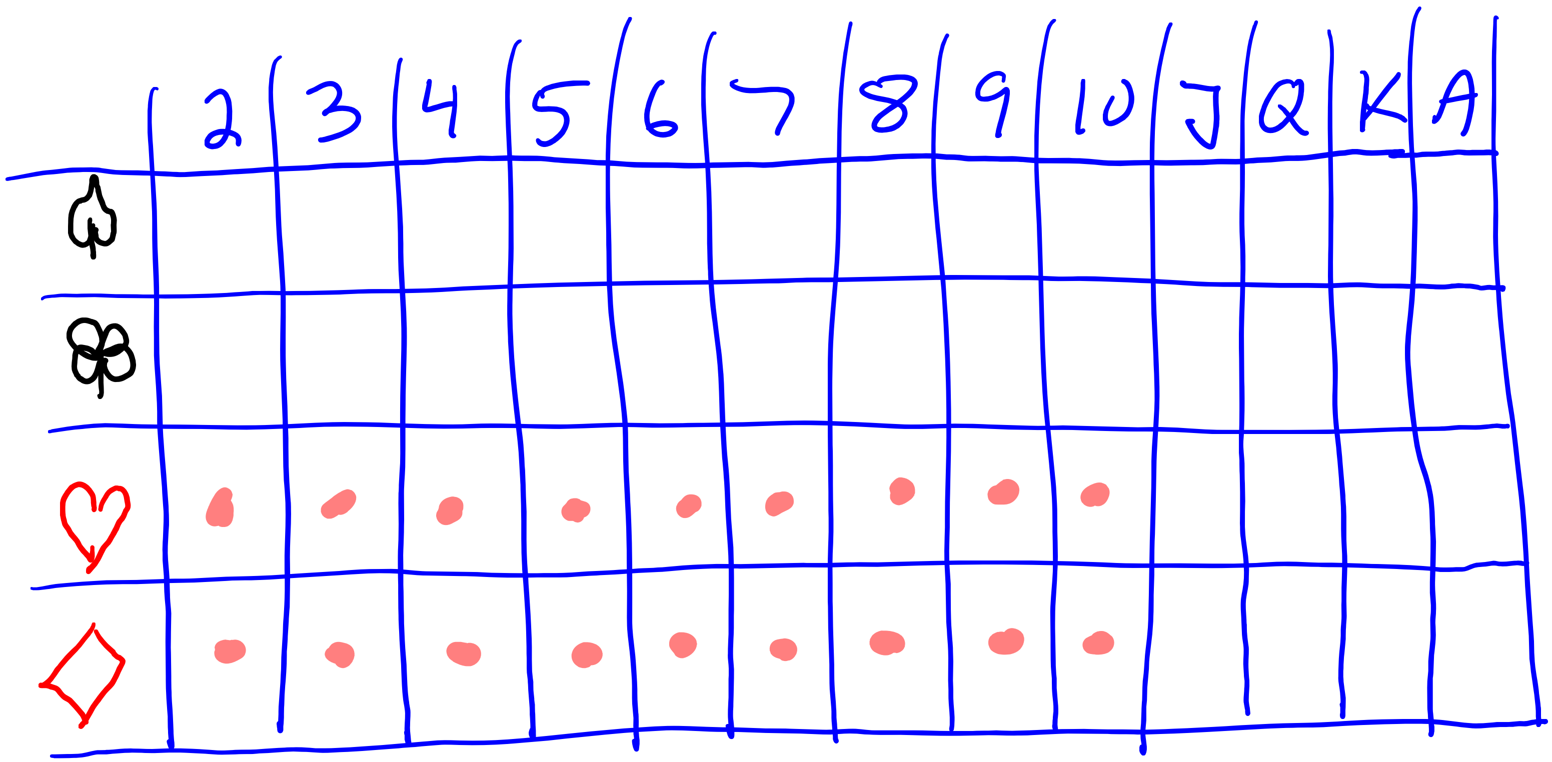
	2	3	4	5	6	7	8	9	10	J	Q	K	A
♠													
♣													
♥													
♦													

$N(S) = 52$

(b) Define E to be the event that the chosen card is red and is not a face card.

Illustrate event E on the illustration of the sample space from **(a)**.

Then, find $N(E)$ and $P(E)$.



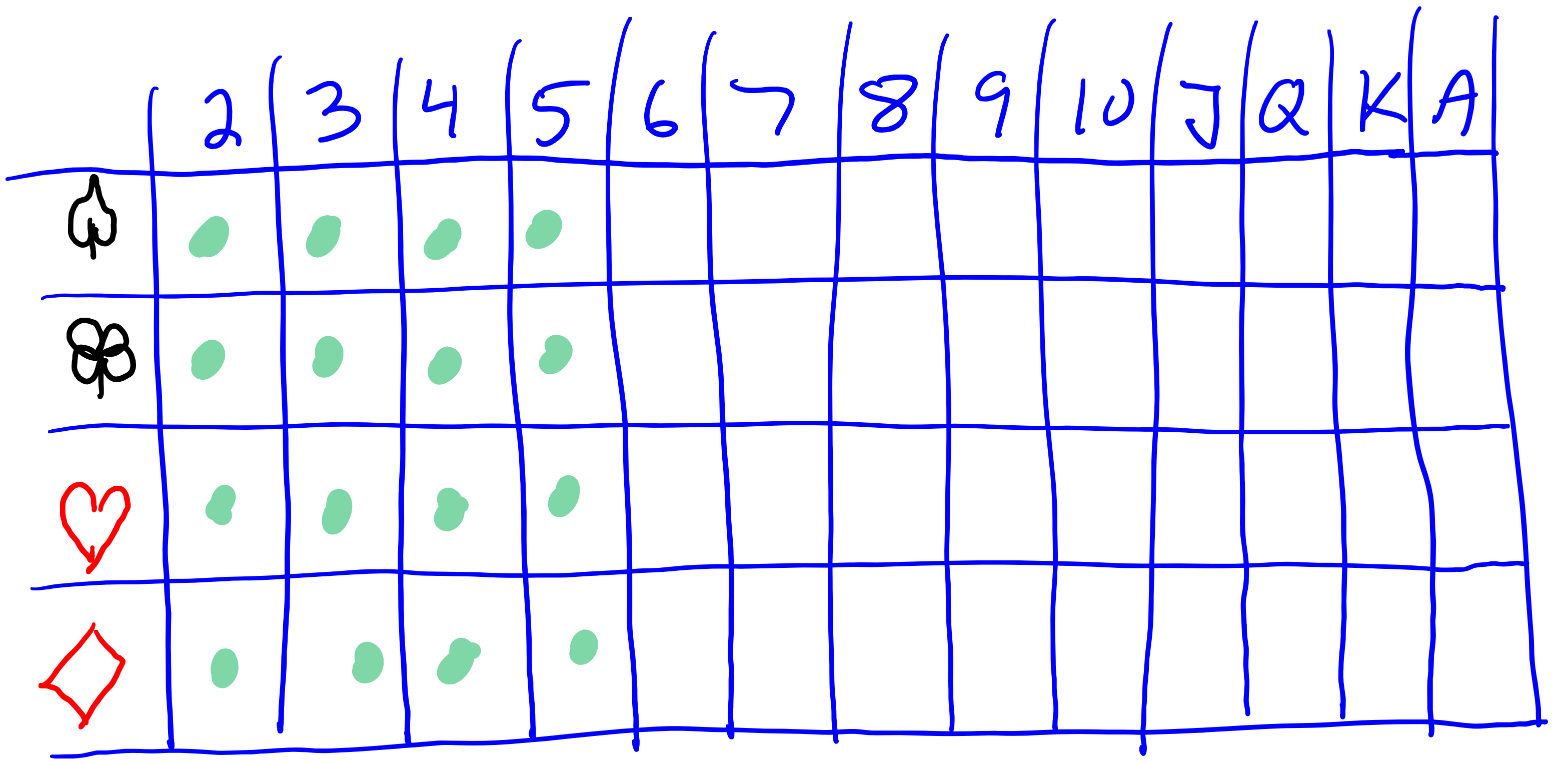
$N(E) = 18$

$$P(E) = \frac{N(E)}{N(S)} = \frac{18}{52} = \frac{9}{26}$$

(c) Define F to be the event that the denomination is at most 5 (counting aces high).

Illustrate event F on the illustration of the sample space from **(a)**.

Then, find $N(F)$ and $P(F)$.



$N(F) = 16$

$P(F) = \dfrac{N(F)}{N(S)} = \dfrac{16}{56} = \dfrac{2}{7}$

Rolling dice

A pair of dice is rolled and the numbers are recorded. Each die is a cube with six sides. One die is blue and one die is red. As with the previous example involving drawing a card from a deck, the sample space can be listed explicitely in set notation, but it is very tedious to write and to read

{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,
41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66}

The same goes for a table showing all of the elements of the sample space S explicitly

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

But as with the playing card example, it is possible to illustrate the sample space using a hand-drawn table that does not take so long to draw and that is also very useful.

.

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

But what if there are three or more dice? Then the sample space can't be so simply illustrated with a table of cells, but some kind of table presentation of the sample space may still be possible and may more helpful than the set notation presentation. (We encountered this in **[Example 1]**, when discussing a coin being flipped four times.)

[Example 3] (similar to 9.1#16,17) Rolling a die with colored faces

Two faces of a six-sided die are painted red, two are painted green, and two are painted blue. The die is rolled three times, and the colors that appear face up on each roll are recoreded. For instance, *RGB* would denote that the colors that appear face up on the first, second, and third roll were *red, green, blue*.

(a) List the elements the sample space S in a table. Find $N(S)$.

RRR RRG RRB	GRR GRG GRB	BRR BRG BRB
RGR RGG RGB	GGR GGG GGB	BGR BGG BGB
RBR RBG RBB	GBR GBG GBB	BBR BBG BBB

$N(S) = 27$

(b) Define E to be the event that all three rolls produce different colors.

Illustrate event E on the illustration of the sample space from **(a)**.

List all outcomes in the event in set notation.

Find $N(E)$ and $P(E)$.

$E = \{RGB, RBG, GRB, GBR, BRG, BGR\}$

$N(E) = 6$

$P(E) = \frac{N(E)}{N(S)} = \frac{6}{27} = \frac{2}{9}$

RRR	GRR	BRR
RRG	GRG	BRG
RRB	GRB	BRB
RGR	GGR	BGR
RGG	GGG	BGG
RGB	GGB	BGB
RBR	GBR	BBR
RBG	GBG	BBG
RBB	GBB	BBB

(c) Define F to be the event that two of the colors that appear face up are the same.

Illustrate event F on the illustration of the sample space from **(a)**.

List all outcomes in the event in set notation.

Find $N(F)$ and $P(F)$.

RRR	GRR	BRR
RRG	GRG	BRG
RRB	GRB	BRB
RGR	GGR	BGR
RGG	GGG	BGG
RGB	GGB	BGB
RBR	GBR	BBR
RBG	GBG	BBG
RBB	GBB	BBB

$N(F) = 18$

$$P(F) = \frac{N(F)}{N(S)} = \frac{18}{27} = \frac{2}{3}$$

Choosing multiple items with replacement versus without replacement

[Example 4](Similar to 9.1#18,19) (drawing *with replacement*)

An urn contains two red balls labeled R_1 and R_2, and two green balls labeled G_1 and G_2, and two blue balls labeled B_1 and B_2.

- A ball is drawn from the urn, its color recorded.
- **Then the ball is put back into the urn. (*replacement*)**
- Then a ball is again drawn from the urn, its color recorded.

(a) Draw the sample space S using a table and find $N(S)$.

Second Ball

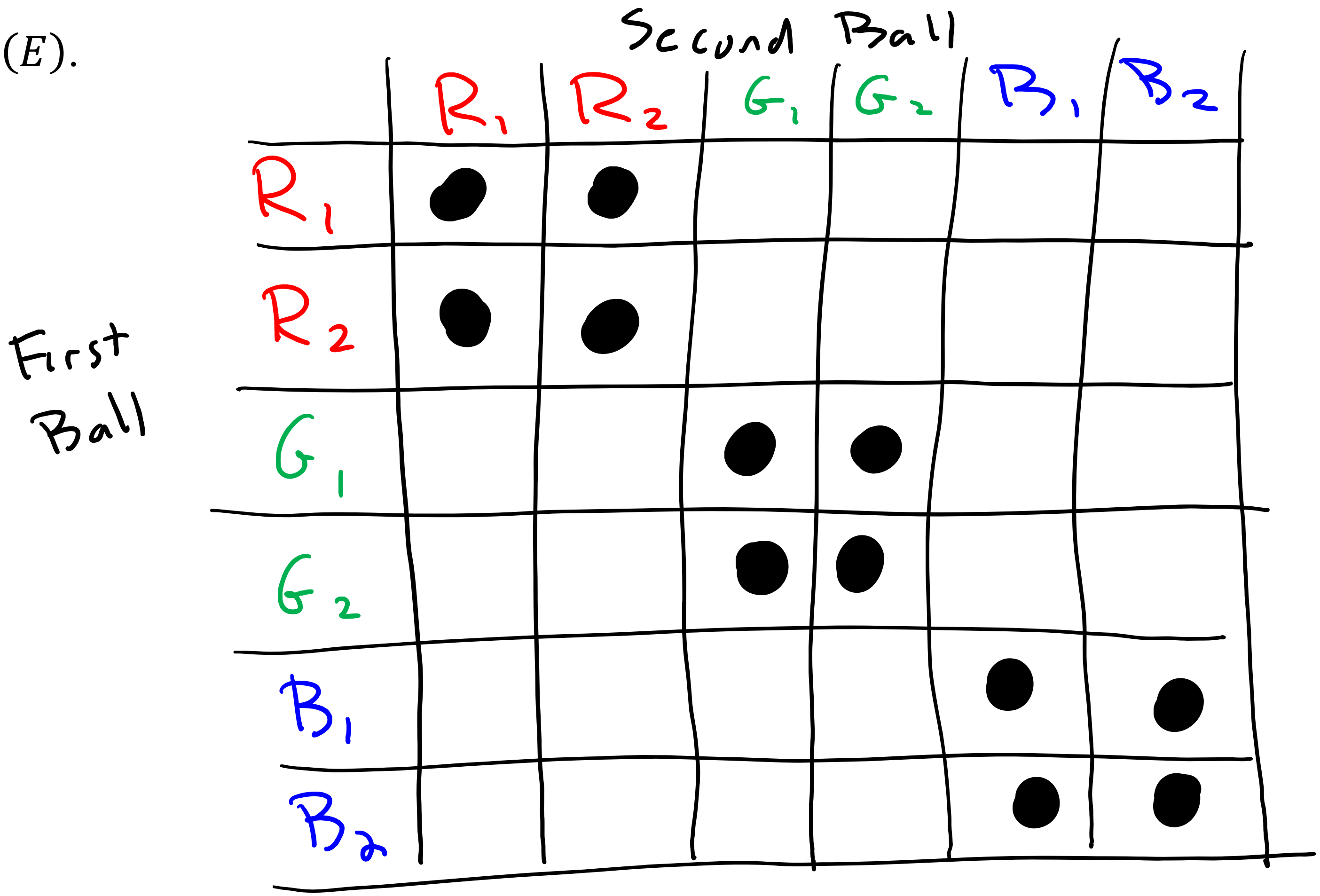
First Ball	R_1	R_2	G_1	G_2	B_1	B_2
R_1						
R_2						
G_1						
G_2						
B_1						
B_2						

$N(S) = 36$

(b) Define E to be the event that the two drawn balls are the same color.

Illustrate event E on the illustration of the sample space from **(a)**.

Find $N(E)$ and $P(E)$.



$$N(E) = 12$$

$$P(E) = \frac{N(E)}{N(S)} = \frac{12}{36} = \frac{1}{3}$$

(c) Define F to be the event that the two drawn balls are not the same color.

Illustrate event F on the illustration of the sample space from **(a)**.

Find $N(F)$ and $P(F)$.

Second Ball

First Ball

	R_1	R_2	G_1	G_2	B_1	B_2
R_1			●	●	●	●
R_2			●	●	●	●
G_1	●	●			●	●
G_2	●	●			●	●
B_1	●	●	●	●		
B_2	●	●	●	●		

$N(F) = 24$

$$P(F) = \frac{N(F)}{N(S)} = \frac{24}{36} = \frac{2}{3}$$

[Example 5](A variation on the previous example, but drawing *without replacement*)

An urn contains two red balls labeled R_1 and R_2, and two green balls labeled G_1 and G_2, and two blue balls labeled B_1 and B_2.

- A ball is drawn from the urn, its color recorded.
- **The ball is *not* put back into the urn. (*no replacement*)**
- Then a ball is again drawn from the urn, its color recorded.

(a) Draw the sample space S using a table and find $N(S)$.

Second Ball

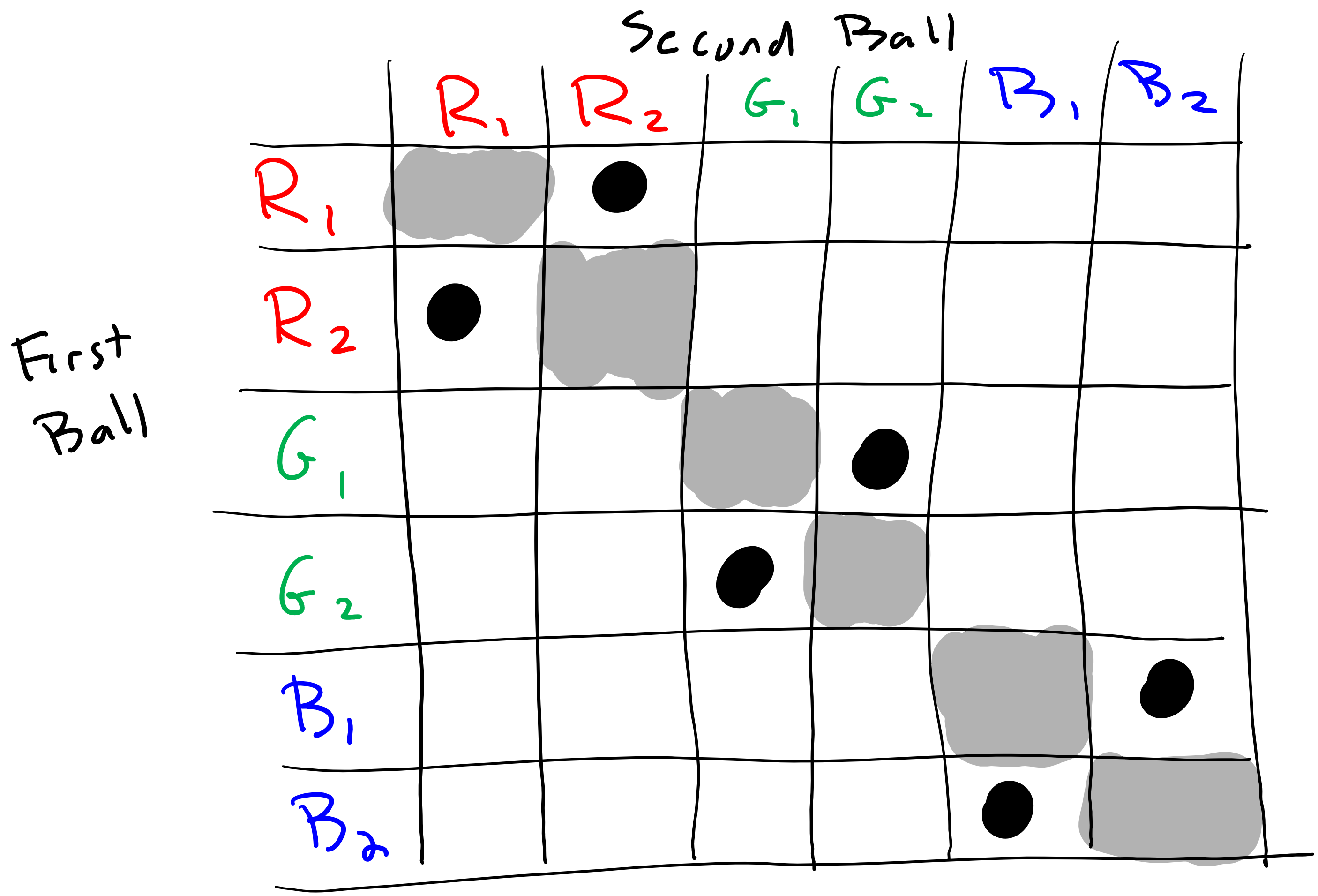
First Ball	R_1	R_2	G_1	G_2	B_1	B_2
R_1						
R_2						
G_1						
G_2						
B_1						
B_2						

$N(S) = 30$

(b) Define E to be the event that the two drawn balls are the same color.

Illustrate event E on the illustration of the sample space from **(a)**.

Find $N(E)$ and $P(E)$.



$N(E) = 6$

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{30} = \frac{1}{5} = 0.2$$

(c) Define F to be the event that the two drawn balls are not the same color.

Illustrate event F on the illustration of the sample space from **(a)**.

Find $N(F)$ and $P(F)$.

Second Ball

First Ball

	R_1	R_2	G_1	G_2	B_1	B_2
R_1			●	●	●	●
R_2			●	●	●	●
G_1	●	●			●	●
G_2	●	●			●	●
B_1	●	●	●	●		
B_2	●	●	●	●		

$N(F) = 24$

$$P(F) = \frac{N(F)}{N(S)} = \frac{24}{30} = \frac{4}{5} = 0.8$$

Counting the Elements of a List

Suppose that you wanted to answer this question:

Question: What is the probability that a randomly chosen 4 digit integer is a multiple of 7?

The approach to a solution would be the following.

- Define the following random process: Choose a 4-digit integer at random. Then the sample space S would be the set of all ~~2~~ 4-digit integers. Find $N(S)$.
- Let E be the event that the randomly chosen ~~2~~ 4-digit integer is a multiple of ~~3~~ 7. Find $N(E)$.
- Compute

$$P(E) = \frac{N(E)}{N(S)}$$

Observe that finding $N(S)$ and $N(E)$ both involve counting the elements of a list.

The book presents a theorem for use when counting lists

> **Theorem 9.1.1 The Number of Elements in a List**
>
> If m and n are integers and $m \leq n$,
>
> then there are $n - m + 1$ integers from m to n inclusive.
>
> It is better to remember this number as $n - (m - 1)$

But I strongly recommend *not* relying on that theorem. Rather, each time you need to count the elements of a list, I recommend that you show the list with an index for each item on the list, and then come up your own formula for the number of items on the list. Figure out a way of presenting the calculation that is clear for you (and for the reader). Once you are done, you may confirm that your results match the result given by the theorem.

[Example 6](Similar to 9.1#21,22) (randomly chosen integers)

(a) A four digit positive integer is chosen at random. Describe the sample space S using set notation. (An ellipsis, …, will be useful unless you want to use up a lot of paper.)

$$S = \{1000, 1001, \ldots, 9998, 9999\}$$

(b) Find $N(S)$. Show the details of your calculation clearly.

$$1, 2, \ldots, 999, \underset{m}{1000}, 1001, \ldots, 9998, \underset{n}{9999}$$

$$N(S) = 9999 - 999 = 9000$$

$$n - (m-1)$$

list

(c) Let E be the event that the integer chosen is a multiple of 7. Describe E using set ~~notation~~.

n	$7n$
1	$7 \cdot 1 = 7$
2	$7 \cdot 2 = 14$
$\vdots$	
142	$7 \cdot 142 = 994$
m 143	$7 \cdot 143 = 1001$
$\vdots$	$\vdots$
n 1428	$7 \cdot 1428 = 9996$
1429	$7 \cdot 1429 = 10{,}003$

(d) Find $N(E)$. Show the details of your calculation clearly.

$$N(E) = 1428 - 142 = 1286$$

$$n - (m - 1)$$

(e) Find $P(E)$. Give an exact answer and a decimal approximation.

$$P(E) = \frac{N(E)}{N(S)} = \frac{1286}{9000} \approx 0.143$$