Video for Homework H09.2 Possibility Trees and the Multiplication Rule

Reading: Section 9.2 Possibility Trees and the Multiplication Rule

Homework: H09.2: 9.2#2,5,10,14,17,22,25,28,32,37,40

Topics:

- **Possibility Trees**
- **The Multiplication Rule**
- **Permutations**

Recall examples from previous video involving using tables to illustrate sample spaces:

Throwing two 6-sided dice. (one blue die, one red die)

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Drawing two balls from an urn containing colored balls.

Drawing with Replacement

	R_1	R_2	G_1	G_2	B_1	B_2
R_1						
R_2						
G_1						
G_2						
B_1						
B_2						

Drawing without Replacement

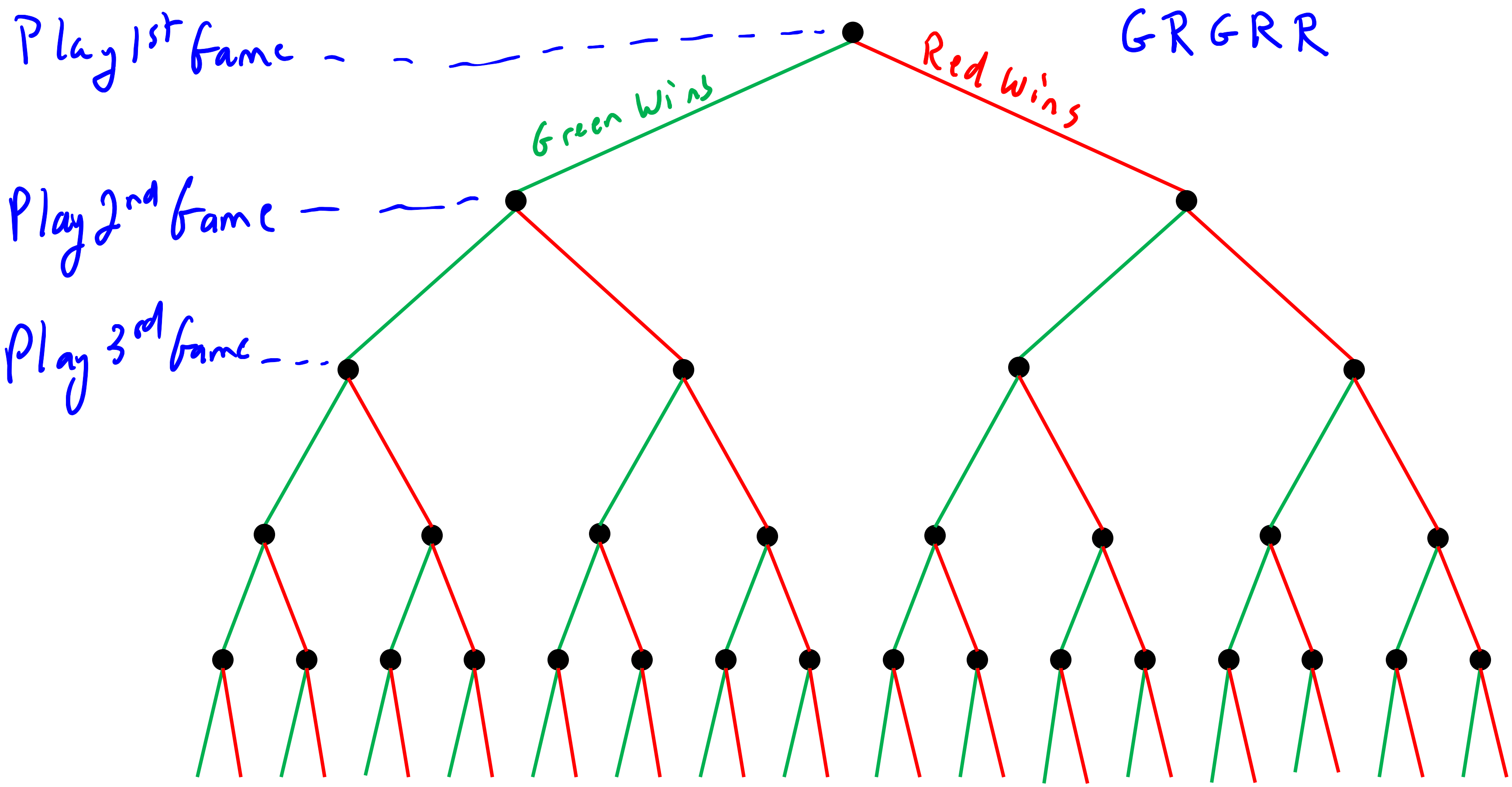
	R_1	R_2	G_1	G_2	B_1	B_2
R_1						
R_2						
G_1						
G_2						
B_1						
B_2						

In some situations, it is not possible to make a table.

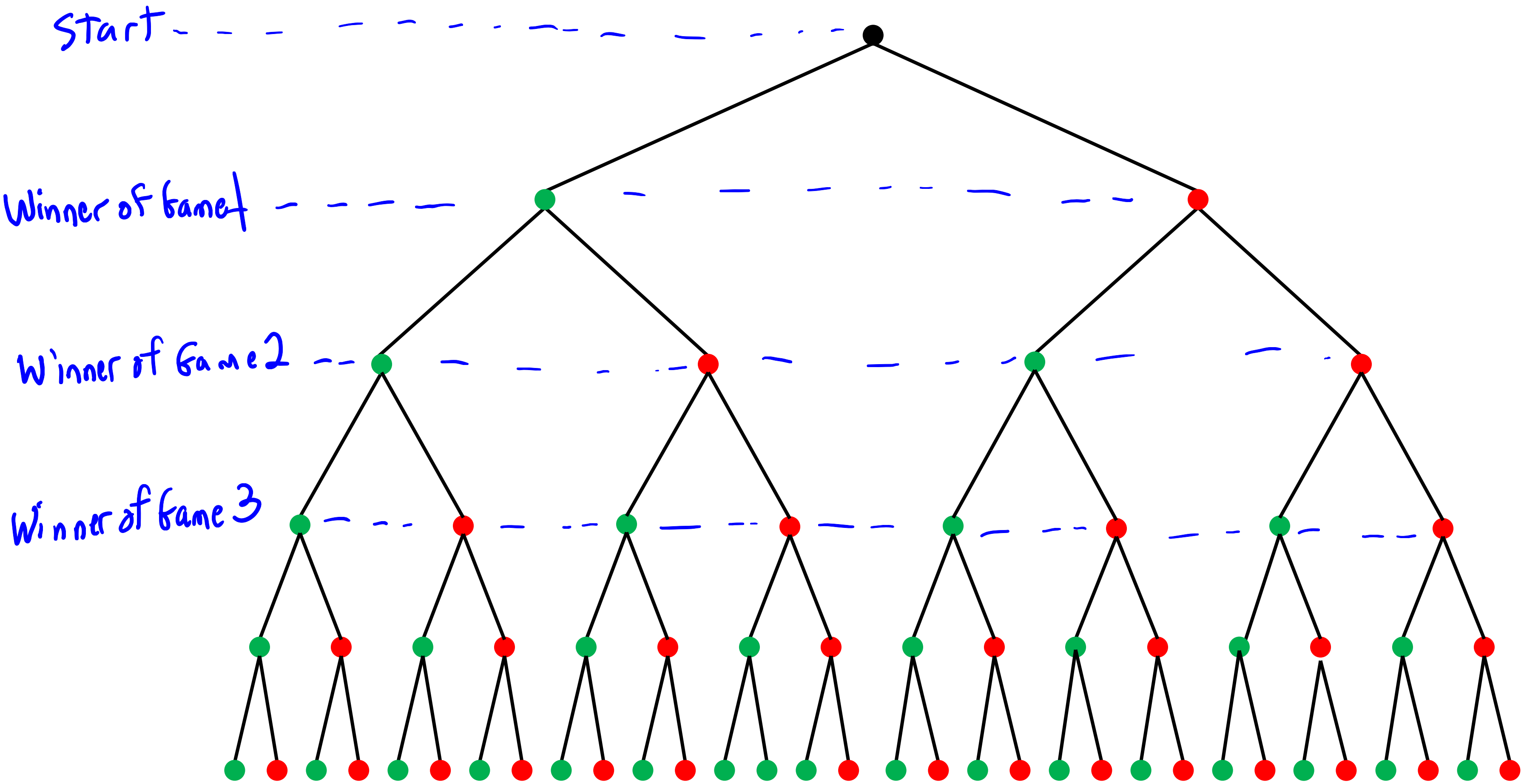
- There may be more steps than can be illustrated in a table
- Or possibilities for later steps may depend on outcomes of earlier steps.

In those cases, knowing how to make a ***Possibility Tree*** is very useful.

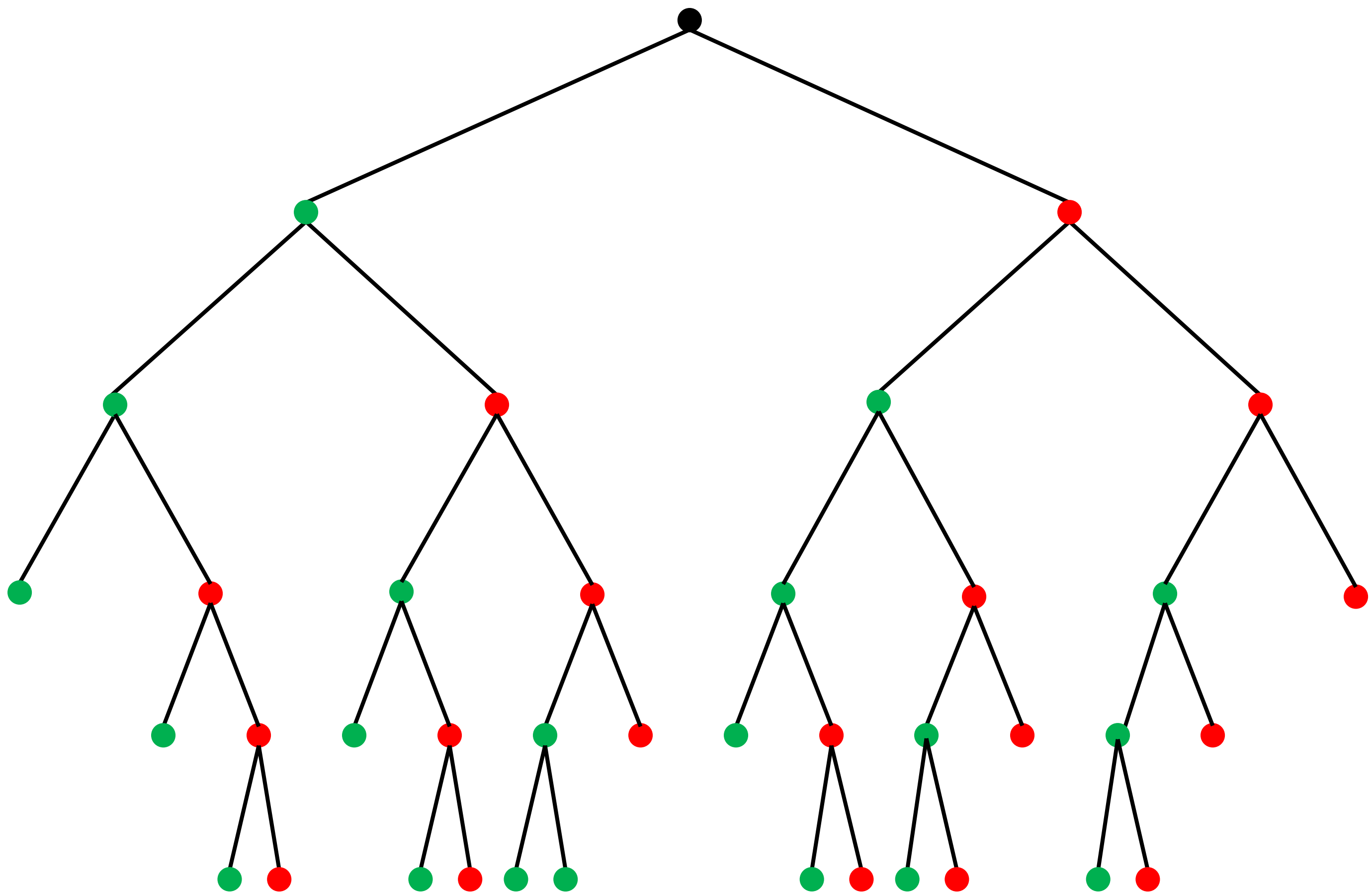
[Example 1](a)(sequencee of 5 games) Consider two baseball teams, the Green Team and the Red Team, playing five games. The diagram below is my preferred way to make a tree diagram. The dots, or *vertices*, represent games to be played. The segments, or *edges*, coming out of the dots represent the winner of the game. Each sequence of 5 games could be represented by a string of 5 letters. Note that the total number of 5 game sequences is $32 = 2^5$.



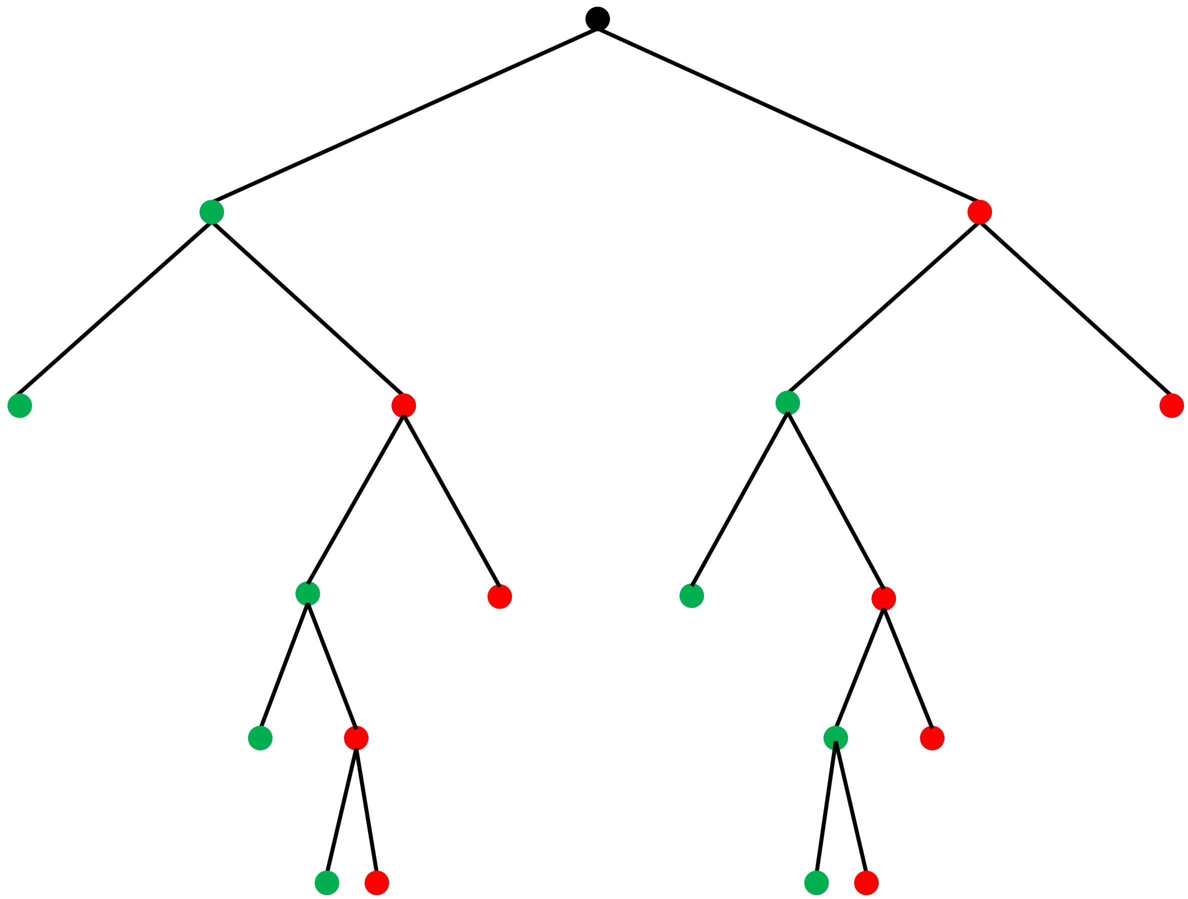
The diagram below is the way that the author of our book makes tree diagram. The *vertices* represent the outcome of the experiments. In this diagram, a dot represents the winner of a game. The edges don't really play a role. I don't like this way of making trees, but it is what the author does, so we will make trees this way. (Note that the trees in the book (and in WebAssign) are oriented horizontally.)



[Example 1](b)(best 3 of 5 games) Now consider a series where the winner is the first team to win 3 games. The tree diagram would be as shown. Notice that there are some scenarios where only 3 or 4 games are played. Notice that there are 16 possible sequences of games that lead to a winner. The book author would say that there are 16 ways for the series to be played.

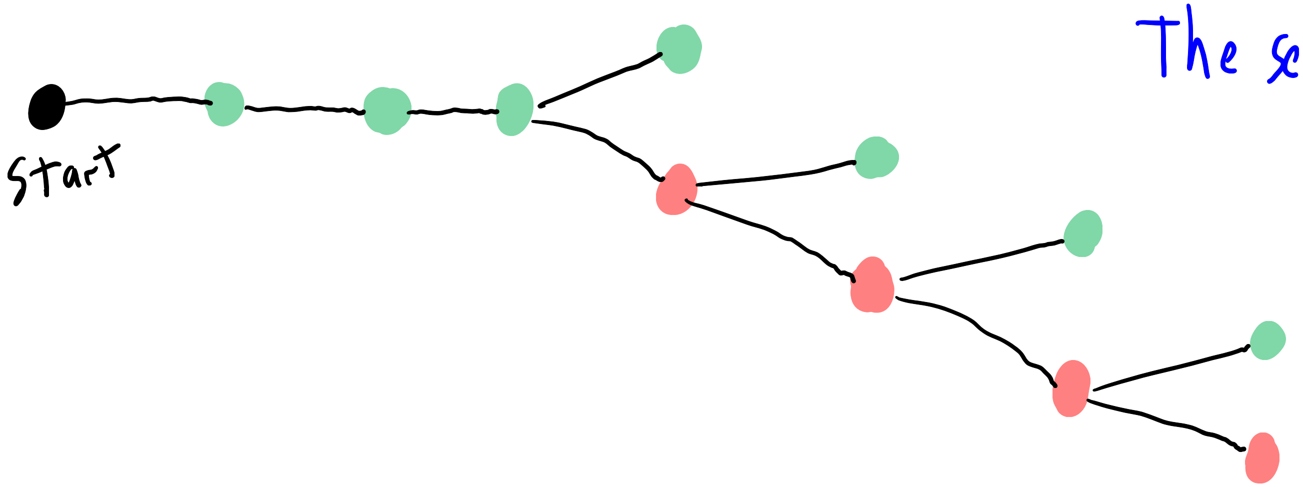


[Example 1](c)(First to get 2 games in a row or win 3 games) Now consider a series where the winner is the first team to win 2 games in a row or 3 games total. The tree diagram would be as shown. Notice that there are 8 possible sequences of games that lead to a winner.



[Example 2] In the World Series, the first team to win four games wins the series.

(a) Suppose the Green Team wins the first three games. How may ways can the series be completed? Make a possibility tree to illustrate.



The series can be completed 5 ways

(b) If all outcomes are equally likely and the Green Team has one the first 3 games, what is the probability that the Red Team will win the Series?

$$P(\text{Red wins}) = \frac{N(\text{Red Wins})}{N(\text{total number of ways})} = \frac{1}{5} = 0.2$$

The Multiplication Rule

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed],*

$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed],*

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

[Example 3] (Similar to 9.2#14) Ohio license plates have the form

AA##AA

where the As stand for letters and the # stand for digits.

(a) How many possible license plates are possible?

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

A_1 A_2 $\#_1$ $\#_2$ A_3 A_4

task number	task description	number of ways
1	Choose A_1	$n_1 = 26$
2	Choose A_2	$n_2 = 26$
3	Choose $\#_1$	$n_3 = 10$
4	Choose $\#_2$	$n_4 = 10$
5	Choose A_3	$n_5 = 26$
6	Choose A_4	$n_6 = 26$

total number of ways is $n = n_1 \cdot n_2 \cdots n_6 = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 26 \cdot 26 = 45{,}697{,}600$

(b) How many possible plates if no character can be used more than once?

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$

$A_1 \quad A_2 \quad \#_1 \quad \#_2 \quad A_3 \quad A_4$

task number	task description	number of ways
1	Choose A_1	$n_1 = 26$
2	Choose A_2	$n_2 = 25$
3	Choose $\#_1$	$n_3 = 10$
4	Choose $\#_2$	$n_4 = 9$
5	Choose A_3	$n_5 = 24$
6	Choose A_4	$n_6 = 23$

total number of ways is $n = n_1 \cdot n_2 \cdots n_6 = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 24 \cdot 23 = 32{,}292{,}000$

(c) How many possible plates could begin with *OU*?

$$\underset{A_1}{\underline{O}}\quad \underset{A_2}{\underline{U}}\quad \underset{\#_1}{\underline{\quad}}\quad \underset{\#_2}{\underline{\quad}}\quad \underset{A_3}{\underline{\quad}}\quad \underset{A_4}{\underline{\quad}}$$

task number	task description	number of ways
1	choose A_1	$n_1 = 1$
2	Choose A_2	$n_2 = 1$
3	choose $\#_1$	$n_3 = 10$
4	Choose $\#_2$	$n_4 = 10$
5	choose A_3	$n_5 = 26$
6	Choose A_4	$n_6 = 26$

total number of ways is $n = n_1 \cdot n_2 \cdot \cdots n_6 = 1 \cdot 1 \cdot 10 \cdot 26 \cdot 26 = 67{,}600$

[Example 4] (similar to 9.2 #32,33,34) Number of Ways to Seat a Group

Seven people are sitting together in a row at a soccer game.

(a) How many ways can they be seated together in the row?

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$
1 2 3 4 5 6 7

task number	description	number of ways
1	choose person for seat 1	$n_1 = 7$
2	choose person for seat 2	$n_2 = 6$
3	Choose person for seat 3	$n_3 = 5$
4	Choose person for seat 4	$n_4 = 4$
5	Choose person for seat 5	$n_5 = 3$
6	choose person for seat 6	$n_6 = 2$
7	Choose person for seat 7	$n_7 = 1$

Total number of ways $n = n_1 \cdot n_2 \cdots n_7 = 7(6)(5)\cdots(2)(1) = 7! = 5040$

(b) Suppose it is a family that wants to sit with the parents on the end (with the mom on the left) and the five kids in the middle. How many ways can the family be seated?

M __ __ __ __ __ D

1 2 3 4 5 6 7

task number	description	number of ways
1	choose person for seat 1	$n_1 = 1$
2	choose person for seat 2	$n_2 = 5$
3	Choose person for seat 3	$n_3 = 4$
4	Choose person for seat 4	$n_4 = 3$
5	Choose person for seat 5	$n_5 = 2$
6	Choose person for seat 6	$n_6 = 1$
7	Choose person for seat 7	$n_7 = 1$

Total number of ways $n = n_1 \cdot n_2 \cdots n_7 = 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 5! = 120$

(c) Suppose it is a family that wants to sit with the parents on the end (with no preference for who sits on the left) and the five kids in the middle. How many ways can the family be seated?

P _ _ _ _ _ P

1 2 3 4 5 6 7

task number	description	number of ways
1	choose person for seat 1	$n_1 = 2$
2	choose person for seat 2	$n_2 = 5$
3	Choose person for seat 3	$n_3 = 4$
4	Choose person for seat 4	$n_4 = 3$
5	Choose person for seat 5	$n_5 = 2$
6	Choose person for seat 6	$n_6 = 1$
7	Choose person for seat 7	$n_7 = 1$

Total number of ways $n = n_1 \cdot n_2 \cdots n_7 = 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 2 \cdot 5! = 240$

(d) Suppose it is a family, and the parents want to be seated together (with no preference for who sits on the left). How many ways can the family be seated?

__ __ __ __ __ __ __
1 2 3 4 5 6 7

task number	description	number of ways
1	choose seat for left-most parent	$n_1 = 6$
2	choose left parent	$n_2 = 2$
3	choose left most kid	$n_3 = 5$
4	choose choose next kid	$n_4 = 4$
5	choose choose next kid	$n_5 = 3$
6	choose choose next kid	$n_6 = 2$
7	choose choose right most kid	$n_7 = 1$

Total number of ways $n = n_1 \cdot n_2 \cdots n_7 = 6 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 12 \cdot 5! = 1440$

[Example 5] (Similar to 9.2#17)

(a) How many integers are there from 100 through 999?

$1, 2, 3, \cdots 99, \quad \underset{m}{100}, 101, \cdots, \underset{n}{999}$

number of integers $= 999 - 99 = 900$

$\qquad\qquad\qquad\quad n \;\; - (m-1)$

Or, we could approach this by imagining choosing digits

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}$

$d_2 \quad d_1 \quad d_0$

task number	description	number of ways
0	Choose digit d_0	$n_0 = 10$
1	choose digit d_1	$n_1 = 10$
2	choose digit d_2	$n_2 = 9$

total number of ways $= n = n_0 \cdot n_1 \cdot n_2 = 10 \cdot 10 \cdot 9 = 900$

(b) How many integers from 100 through 999 are *even*?

Approach this by imagining choosing digits

$\overline{\quad} \; \overline{\quad} \; \overline{\quad}$
$d_2 \quad d_1 \quad d_0$

task number	description	number of ways
0	Choose digit d_0	$n_0 = 5$
1	choose digit d_1	$n_1 = 10$
2	choose digit d_2	$n_2 = 9$

total number of ways $= n = n_0 \cdot n_1 \cdot n_2 = 5 \cdot 10 \cdot 9 = 450$

(c) How many integers from 100 through 999 have *distinct digits*?

Approach this by imagining choosing digits

$\overline{d_2}\ \overline{d_1}\ \overline{d_0}$

task number	description	number of ways
1	Choose digit d_2	$n_0 = 9$
2	choose digit d_1	$n_1 = 9$
3	choose digit d_0	$n_2 = 8$

total number of ways $= n = n_0 \cdot n_1 \cdot n_2 = 9 \cdot 9 \cdot 8 = 648$

(d) How many integers from 100 through 999 are *even* and have *distinct digits*?

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}$

$d_2 \quad d_1 \quad d_0$

Case I d_2 is odd

task	description	number of ways
1	choose d_2	$n_1 = 5$
2	choose d_0	$n_2 = 5$
3	choose d_1	$n_3 = 8$

total

$n = n_1 \cdot n_2 \cdot n_3$
$= 5 \cdot 5 \cdot 8$
$= 200$

Case II d_2 is even

task	description	number of ways
1	choose d_2	$n_1 = 4$
2	choose d_0	$n_2 = 4$
3	choose d_1	$n_3 = 8$

total

$n = n_1 \cdot n_2 \cdot n_3$
$= 4 \cdot 4 \cdot 8$
$= 128$

total number: $n = 200 + 128 = 328$

(e) What is the probability that a randomly chosen ~~four~~ three-digit integer has *distinct digits*?

Event E

$$P(E) = \frac{N(E)}{N(S)} = \frac{648}{900} = 0.72$$

(f) What is the probability that a randomly chosen ~~four~~ three-digit integer is *even* and has *distinct digits*?

Event F

$$P(F) = \frac{N(F)}{N(S)} = \frac{328}{900} = 0.36\overline{4}$$

[Example 6] (Similar to 9.2#25) Counting iterations of a loop.

Consider the following algorithm segment

for $i := 5$ **to** 50

for $j := 10$ **to** 20

[Statements in body of inner loop.

None contains branching statements that lead outside the loop.]

next j

next i

How many times will the innermost loop be iterated when the algorithm segment is run?

The innermost loop will be run once for each pair (i, j)

Such that $5 \leq i \leq 50$ and $10 \leq j \leq 20$

The number of pairs will be

Number of choices for i $\cdot$ Number of choices for j

$(50-(5-1)) \cdot (20-(10-1))$

$46 \cdot 11$

506

[Example 7](Similar to 9.2 # 21,22) Counting Relations and Functions

Suppose A is a set with m elements and B is a set with n elements.

(a) How many elements are in the cartesian product $A \times B$?

Solution: $A \times B = \{ (a, b)$ such that $a \in A$ and $b \in B \}$

Consider the task of building an ordered pair $(\underline{\ \ }_a, \underline{\ \ }_b)$

task number	description	number of ways
1	choose a	$K_1 = m$
2	choose b	$K_2 = n$

total number of ways $K = K_1 \cdot K_2 = m \cdot n$

So there are $m \cdot n$ ordered pairs of the form (a, b) in the cartesian product $A \times B$.

(b) How many *relations* are there from A to B?

Recall a relation from A to B is a subset $R \subseteq A \times B$

So the number of relations will be the number of subsets of $A \times B$

Consider the task of building a subset $R \subseteq A \times B$

task number	description	number of ways
1	decide whether $(a,b)_1 \in R$	$2 = k_1$
2	decide whether $(a,b)_2 \in R$	$2 = k_2$
$\vdots$	$\vdots$	$\vdots$
$m \cdot n$	decide whether $(a,b)_{m \cdot n} \in R$	$2 = k_{m \cdot n}$

The total number of ways will be

$$k = k_1 \cdot k_2 \cdots k_{m \cdot n} = \underbrace{2 \cdot 2 \cdot \cdots 2}_{m \cdot n \text{ factors of } 2} = 2^{m \cdot n}$$

(c) How many *functions* are there from A to B?

A function $f: A \to B$ takes as input an element of A and produces as output an element of B.

task number	description	number of ways
1	Choose output $f(a_1) \in B$	$k_1 = n$
2	Choose output $f(a_2) \in B$	$k_2 = n$
$\vdots$	$\vdots$	$\vdots$
m	Choose output $f(a_m) \in B$	$k_m = n$

The total number of ways will be

$$k = k_1 \cdot k_2 \cdot \cdots \cdot k_m = \underbrace{n \cdot n \cdot \cdots \cdot n}_{m \text{ factors of } n} = n^m$$

(d) How many *one-to-one functions* are there from A to B?

no output can occur more than once

task number	description	number of ways
1	choose output $f(a_1) \in B$	$k_1 = n$
2	choose output $f(a_2) \in B$	$k_2 = n-1$
$\vdots$	$\vdots$	$\vdots$
m	choose output $f(a_m) \in B$	$k_m = n-(m-1)$

The total number of ways will be

$$k = k_1 \cdot k_2 \cdot \cdots \cdot k_m = n(n-1)\cdots(n-(m-1)) = \frac{n!}{(n-m)!}$$

(e) How many *functions* are there from a set with 7 elements to a set with 3 elements?

.

$m=7$ $n=3$

Number of functions is

$$n^m = 3^7 = 2187$$

Permutations:

> **Definition of Permutations**
>
> A ***permutation*** of a set is a *choice of an ordering* of the elements of the set.
>
> An ***r-permutation*** of a set of n elements is an ordered selection of r elements taken from the set of n elements. (So if a set has n elements, then a *permutation* of the set is the same thing as an *n-permutation* of the set.)
>
> The number of *r-permutations* of a set of n elements is denoted $P(n, r)$.

[Example 8] (similar to 9.2#35,36)

(a) Give examples of three permutations of the set {a,b,c,d,e,f,g}

abcdefg , gfedcba , bacdefg

(b) Give examples of 3-permutations of the set {a,b,c,d,e,f,g}

abc , dgc

(c) For the set {a,b,c,d,e,f,g}, how many permutations are there?

__ __ __ __ __ __ __
1 2 3 4 5 6 7

task number	description	number of ways
1	choose letter 1	$n_1 = 7$
2	" " 2	$n_2 = 6$
3	" " 3	$n_3 = 5$
4	" " 4	$n_4 = 4$
5	" " 5	$n_5 = 3$
6	" " 6	$n_6 = 2$
7	" " 7	$n_7 = 1$

total number of ways is

$$n = n_1 \cdot n_2 \cdots n_7$$
$$= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 7!$$
$$= 5040$$

(d) For the set {a,b,c,d,e,f,g}, how many 3-permutations are there?

__ __ __
1 2 3

task number	description	number of ways
1	choose letter 1	$n_1 = 7$
2	" " 2	$n_2 = 6$
3	" " 3	$n_3 = 5$

total number of ways is $n = n_1 \cdot n_2 \cdot n_3 = 7 \cdot 6 \cdot 5 = 210$

$$= \frac{7!}{4!}$$

Counting the number of permutations

The methods used to find the answers to [Example 8](c)(d) are the same used to prove the following two theorems.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Theorem 9.2.3

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \quad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$

[Example 9] Computing Number of Permutations (Similar to 9.2#37)

Find the following quantities. (Show the calculations clearly.) $P(n,r) = \frac{n!}{(n-r)!}$

$(\boldsymbol{a})\ P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot\cancel{2\cdot1}}{\cancel{2\cdot1}} = 5\cdot4\cdot3 = 60$

$(\boldsymbol{b})\ P(5,1) = \frac{5!}{(5-1)!} = \frac{5!}{4!} = \frac{5\cdot\cancel{4\cdot3\cdot2\cdot1}}{\cancel{4\cdot3\cdot2\cdot1}} = 5$

$(\boldsymbol{c})\ P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1} = 120$

$(\boldsymbol{d})\ P(5,0) = \frac{5!}{(5-0)!} = \frac{5!}{5!} = 1$

[Example 10] 9.2#42) Prove $\forall n \geq 3\big(P(n+1,3) - P(n,3) = 3P(n,2)\big)$.

Left side of the predicate

$$P(n+1,3) - P(n,3) = \frac{(n+1)!}{((n+1)-3)!} - \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n-2)!} - \frac{n!}{(n-3)!} =$$

$$= \frac{(n+1)(n)(n-1)\cancel{(n-2)\cdots(2)(1)}}{\cancel{(n-2)\cdots(2)(1)}} - \frac{n(n-1)(n-2)\cancel{(n-3)\cdots(2)(1)}}{\cancel{(n-3)\cdots(2)(1)}}$$

$$= (n+1)\underbrace{n(n-1)} - \underbrace{n(n-1)}(n-2)$$

$$= \underbrace{n(n-1)}\big((\cancel{n}+1) - (\cancel{n}-2)\big) = n(n-1)(3)$$

Right side of the Predicate

$$3P(n,2) = 3 \cdot \frac{n!}{(n-2)!} = \frac{3 \cdot n \cdot (n-1)\cancel{(n-2)\cdots(2)(1)}}{\cancel{(n-2)\cdots(2)}} = 3 \cdot n(n-1)$$

Conclusion Since left side = right side the predicate is true.