Video for Homework H09.3 Counting Elements of Disjoint Sets: The Addition Rule

Reading: Section 9.3 Counting Elements of Disjoint Sets: The Addition Rule

Homework: H09.3: 9.3#7,9,17,21,22,24,34,36

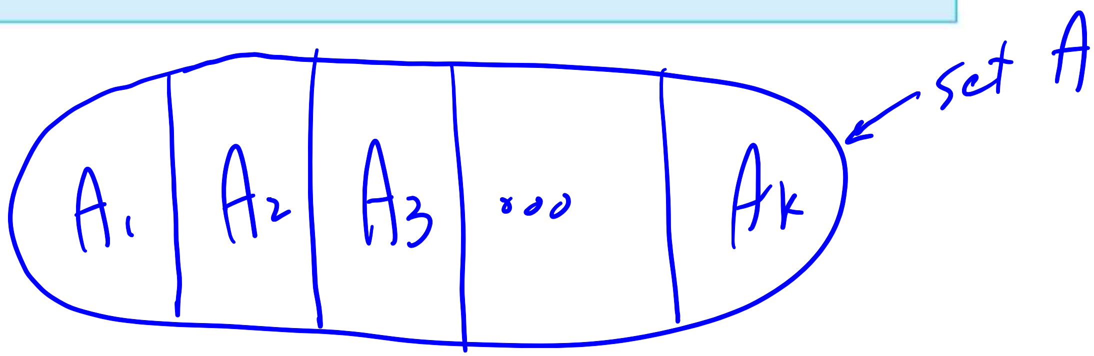
Topics:

- **The Addition Rule**
- **The Difference Rule**
- **The Complement Rule for Probability**
- **The Inclusion/Exclusion Rule**

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then

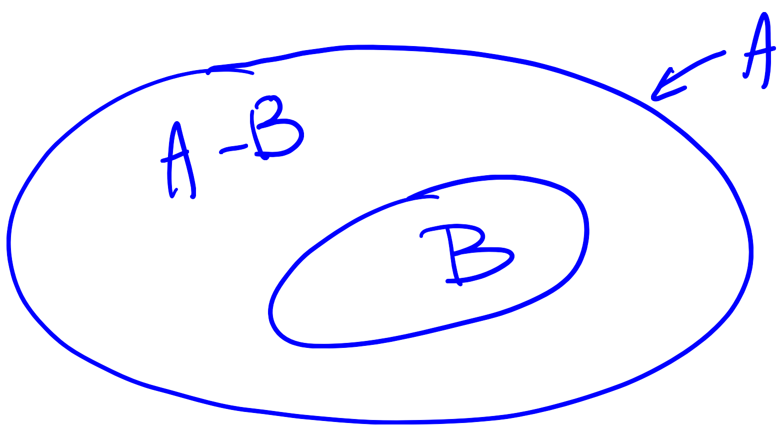
$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$



Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$



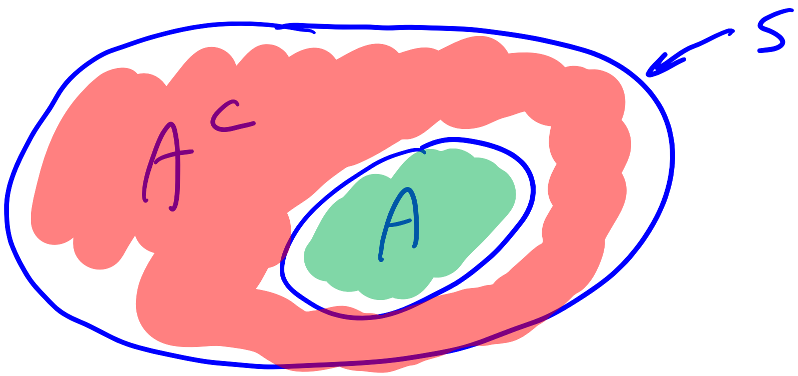
$N(A) = N(A-B) + N(B)$ by Addition Rule

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A),$$

where $A^c = S - A$, the complement of A in S.



$$P(A^c) = \frac{N(A^c)}{N(S)} = \frac{N(S) - N(A)}{N(S)} = \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} = 1 - P(A)$$

Difference Law

[Example 1] (similar to 9.1#17)

(a) How many strings of eight hexadecimal digits do not have any repeated digits?

hexadecimal digits $\underbrace{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}_{16 \text{ characters}}$

_ _ _ _ _ _ _ _
1 2 3 4 5 6 7 8

task	description	number of ways
#1	Choose character for blank #1	$n_1 = 16$
#2	" " " " #2	$n_2 = 15$
#3	" " " " #3	$n_3 = 14$
⋮		⋮
#8	" " " " #8	$n_8 = 9$

$$n = n_1 \cdot n_2 \cdots n_8 = 16(15)(14)\cdots(9) = \frac{16!}{8!} = 518{,}918{,}400$$

(b) How many strings of eight hexadecimal digits have at least one repeated digit?

Let S = the set of all strings of 8 hexadecimal digits

Let A = the set that have no repeated digits

then $S - A$ = the set that have at least one repeated digit.

$$N(S-A) = \underbrace{N(S)} - \underbrace{N(A)}$$

we counted set A in question (a)

We need to count $N(S)$.

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

task #1: choose character for blank #1 $n_1 = 16$

#2 " " " " #2 $n_2 = 16$

⋮ ⋮

#8 " " " " #8 $n_8 = 16$

So $n = n_1 \cdot n_2 \cdot \cdots \cdot n_8 = 16 \cdot 16 \cdot \cdots 16 = 16^8 = 4{,}294{,}967{,}296 = N(S)$

So $N(S-A) = N(S) - N(A) = \cdots = 3{,}776{,}048{,}896$

(c) What is the probability that a randomly chosen string of eight hexadecimal digits has at least one repeated digit?

$$P(S-A) = \frac{N(S-A)}{N(S)} = \frac{3,776,048,896}{4,294,967,296} \approx 0.879$$

[Example 2] (similar to 9.1#7) Password must be from 4 – 6 symbols long, and may include

- upper case letters 26
- lower case letters 26
- digits 10

$\left.\begin{matrix} 26 \\ 26 \\ 10 \end{matrix}\right\}$ 62 total characters to choose from

(a) How many passwords are available if repetition is allowed?

Let A = passwords with 4 characters
B = passwords with 5 characters
C = passwords with 6 characters.
} mutually disjoint

So $N(A \cup B \cup C) = N(A) + N(B) + N(C)$

$= 62^4 + 62^5 + 62^6$

$= 57,731,144,752$

(b) How many passwords have no repeated symbol?

Define D = passwords with $\overline{4}$ characters and no repeated characters

E = " " 5 " " " " "

F = " " 6 " " " " "

These are disjoint so by the Addition Rule

$$N(D \cup E \cup F) = N(D) + N(E) + N(F)$$

$$= 62 \cdot 61 \cdot 60 \cdot 59 + 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 + 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57$$

$$= 45{,}051{,}562{,}200$$

of length 4, 5, 6

(c) How many passwords have at least one repeated symbol?

Passwords with at least one repeated symbol = All passwords − Passwords with no repeats

$= A \cup B \cup C - D \cup E \cup F$

So

Number of Passwords with at least one repeated symbol $= N(A \cup B \cup C) - N(D \cup E \cup F)$

$= 57{,}731{,}144{,}752 - 45{,}051{,}562{,}200$

$= 12{,}679{,}582{,}552$

(d) What is probability that a randomly selected password of length 4,5,6 has at least one repeated symbol?

$$P(\text{at least one repeated symbol}) = \frac{N(\text{at least one repeated symbol})}{N(\text{all passwords})}$$

$$= \frac{12{,}679{,}582{,}552}{57{,}731{,}144{,}752}$$

$$\approx 0.22$$

[Example 3] (similar to 9.1#22) Consider strings of length n over the alphabet $\{a, b, c, d, e\}$

(a) How many such strings contain at least one pair of adjacent characters that are the same?

Let S = set of all strings of length n over the alphabet $\{a,b,c,d,e\}$

A = set of all strings of length n with no pairs of adjacent characters the same.

We are being asked to find $N(S-A)$

Strategy: Find $N(S)$

Find $N(A)$

Compute $N(S-A) = N(S) - N(A)$

To find $N(S)$, consider these tasks

$$\underline{\quad}_1 \ \underline{\quad}_2 \ \underline{\quad}_3 \ \cdots \ \underline{\quad}_n$$

task	description	number of ways
#1	choose letter for blank #1	$n_1 = 5$
#2	" " " " #2	$n_2 = 5$
⋮		⋮
#n	" " " " "	$n_n = 5$

Total number of ways $= n = n_1 \cdot n_2 \cdots n_n = \underbrace{5 \cdot 5 \cdots 5}_{n} = 5^n$

$$N(S) = 5^n$$

To find $N(A)$, consider these tasks

$$\underset{1}{\underline{\quad}}\ \underset{2}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \cdots\ \underset{n}{\underline{\quad}}$$

Task	Description	number of ways
#1	Choose letter for blank #1	$n_1 = 5$
#2	Choose letter for blank #2	$n_2 = 4$
#3	Choose letter for blank #3	$n_3 = 4$
$\vdots$		$\vdots$
#n	Choose letter for blank #n	$n_n = 4$

So the total number of strings is

$$n = n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_n = 5 \cdot \underbrace{4 \cdot 4 \cdot \cdots \cdot 4}_{n-1} = 5 \cdot 4^{n-1}$$

$$N(A) = 5 \cdot 4^{n-1}$$

So $N(S-A) = N(S) - N(A)$

$$= 5^n - 5 \cdot 4^{n-1}$$

(b) If a string of length 7 over the alphabet $\{a, b, c, d, e\}$ is chosen at random, what is the probability that it contains at least one pair of adjacent characters that are the same?

$$P(S-A) = \frac{N(S-A)}{N(S)} = \frac{N(S) - N(A)}{N(S)} = 1 - \frac{N(A)}{N(S)}$$

$$= 1 - \frac{5 \cdot 4^{n-1}}{5^n} \qquad \text{with } n=7$$

$$= 1 - \frac{5 \cdot 4^{7-1}}{5^7} = 1 - \frac{5 \cdot 4^6}{5^7} =$$

$$= 1 - \frac{4^6}{5^6} = 1 - \left(\frac{4}{5}\right)^6 = 1 - (0.8)^6 =$$

$$= 0.737856$$

[Example 4] (similar to 9.1#21) Seven people, denoted by letters a, b, c, d, e, f, g, are seated randomly in 7 seats in a row. What is the probability a and b are not be seated together?

Let S = the set of permutations of the set $\{a, b, c, d, e, f, g\}$

Let A = the permutations that have a, b seated together.

We are being asked for $P(S-A) = P(A^c)$

$$P(A^c) = P(S-A) = \frac{N(S-A)}{N(S)} = \frac{N(S)-N(A)}{N(S)} = 1 - \frac{N(A)}{N(S)}$$

$N(S)$ = number of permutations $= 7!$

Let B = Seatings with a,b together and $\underline{a}$ on left

C = Seatings with a,b together and $\underline{b}$ on left.

Then $A = \underbrace{B \cup C}_{\text{disjoint union.}}$

So $N(A) = N(B) + N(C)$ by Addition Rule.

Find $N(B)$ Consider tasks

_ _ _ _ _ _ _

task	description	number of ways
#1	choose seat for a	$n_1 = 6$
#2	" " " b	$n_2 = 1$
#3	choose letter for left-most empty seat	$n_3 = 5$
#4	choose letter for next seat	$n_4 = 4$
#5		$n_5 = 3$
#6		$n_6 = 2$
#7		$n_7 = 1$

So total number of ways is $N(B) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$

Similarly, $N(C) = 6!$ as well

So $N(A) = N(B) + N(C) = 6! + 6! = 2 \cdot 6!$

So $P(A^c) = 1 - \frac{N(A)}{N(S)} = 1 - \frac{2 \cdot 6!}{7!} =$

$$= 1 - \frac{2 \cdot (\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}{(7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})} = 1 - \frac{2}{7}$$

$$= \frac{5}{7} \approx 0.714$$

[Example 5] (Similar to 9.3#9) Counting iterations of a loop.

Consider the following algorithm segment

for $i := 1$ **to** 1000

for $j := 1$ **to** i

[Statements in body of inner loop.

None contains branching statements that lead outside the loop.]

next j

next i

How many times will the innermost loop be iterated when the algorithm segment is run?

The innermost loop gets iterated once for every pair (i, j),

where $1 \le i \le 1000$ and $1 \le j \le i$

We need to count all those pairs

Set of All Pairs = Pairs with $i=1$ $\cup$ Pairs with $i=2$ $\cup$ Pairs with $i=3$ $\cup \cdots \cup$ Pairs with $i=1000$

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{1000}$$

This is a disjoint union, so

$$N(A) = N(A_1) + N(A_2) + N(A_3) + \cdots + N(A_{1000})$$

$$= \underbrace{1 + 2 + 3 + \cdots + 1000}_{\text{Sum of 1ST 1000 positive integers}}$$

$$= \frac{1000(1000+1)}{2} = 500(1001) =$$

$$= 500{,}500$$

One more new topic for this homework assignment: The Inclusion/Exclusion Rule

Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

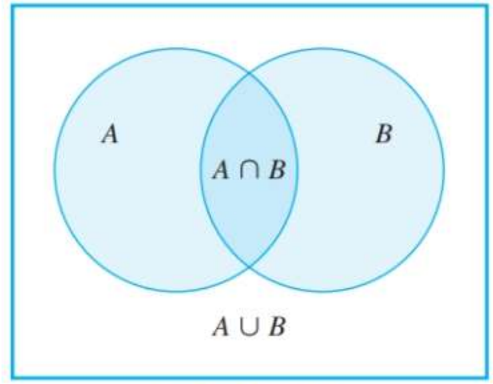
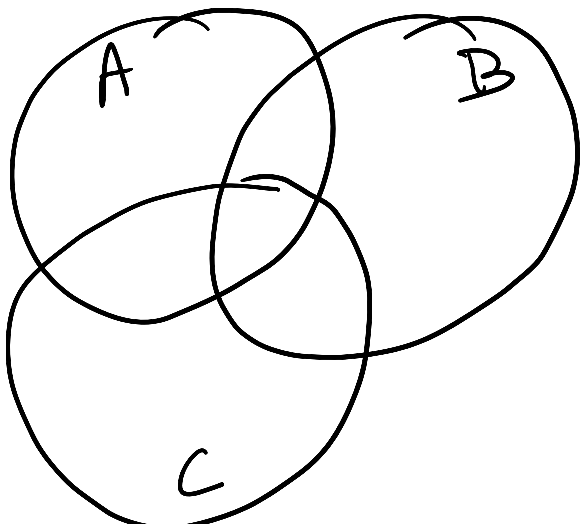


FIGURE 9.3.5



This concept is used is used in the solution to homework problems 9.3#24,34. Book examples 9.3.1, 9.3.6, and 9.3.7 discuss almost identical problems. Because of that, I won't present similar examples in this video.