# Video for Homework H09.5 Counting Subsets of a Set: Combinations

**Reading:** Section 9.5 Counting Subsets of a Set: Combinations

**Homework:** H09.5: 9.5#5,6,10,12,14,16,17,18,20,25

**Topics:**

- **Definition of *r-combination***
- **Counting the *number* of *r-combinations***
- **Examples**

## Recall the Definition of Permutations and Formulas for Number of Permutations

**Definition of Permutations**

A ***permutation*** of a set is a *choice of an ordering* of the elements of the set.

An ***r-permutation*** of a set of $n$ elements is an ordered selection of $r$ elements taken from the set of $n$ elements. (So if a set has $n$ elements, then a *permutation* of the set is the same thing as an *n-permutation* of the set.)

The *number* of *r-permutations* of a set of $n$ elements is denoted $P(n, r)$.

**Theorem 9.2.2**

For any integer $n$ with $n \geq 1$, the number of permutations of a set with $n$ elements is $n!$.

**Theorem 9.2.3**

If $n$ and $r$ are integers and $1 \leq r \leq n$, then the number of $r$-permutations of a set of $n$ elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$

## Section 9.5 is about *Combinations*

**Definition of *r-combination***

An ***r-combination*** of a set of *n* elements is an *unordered* selection of *r* elements taken from the set of *n* elements. That is, it is a subset of *r* elements.

The *number* of *r-combinations* of a set of *n* elements is denoted $C(n, r)$ or $\binom{n}{r}$. This quantity is spoken "*n choose r*".

## Determining a formula for $\boldsymbol{C(n, r)}$

**Natural Question:**

What is the formula for the number of *r-combinations* of a set of *n* elements?

It turns out that we can figure out a formula by considering the relationship between $C(n, r)$ and $P(n, r)$

We already know that the formula for the number of *r-permutations* of a set of *n* elements is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r-1) = \frac{n!}{(n-r)!}$$

We can show how to arrive at this formula using tasks that involve *r-combinations*.

Consider the task of choosing an *r-permutation* as two sub-tasks

number of ways

**Task #1:** Choose a *subset* of *r* elements from the set of *n* elements. $k_1 = C(n,r)$

(That is, choose an *r-combination* of the set of *n* elements.)

**Task #2:** Choose an *ordering* of the elements of that subset of *r* elements $k_2 = r!$

The total number of ways of choosing an r-permutation is found by using the multiplication rule.

$$k = k_1 \cdot k_2 = C(n,r) \cdot r!$$

But we already know that the number must $P(n,r) = \frac{n!}{(n-r)!}$

So $C(n,r) \cdot r! = P(n,r)$

Therefore, $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

**We have just proved Theorem 9.5.1**

**Theorem 9.5.1 Computational Formula for $\binom{n}{r}$**

The number of subsets of size $r$ (or $r$-combinations) that can be chosen from a set of $n$ elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where $n$ and $r$ are nonnegative integers with $r \le n$.

Many of the homework problems on the homework set H09.5 involve problems very similar to book examples. I don’t intend to present similar examples in this video, because I want you to read those book examples very carefully.

But some of the homework exercises involve significant variations from anything presented in the book and have tricky solutions. And some of the exercises, though not tricky, do warrant some discussion. For that reason, I will present examples similar to four of the homework exercises.

**[Example 1] (similar to 9.5#5)** Find the following quantities.

(*a*) $\binom{9}{6} = C(9,6) = \frac{9!}{6!(9-6)!} = \frac{9!}{6!\,3!} = \frac{9\cdot 8\cdot 7\cdot \cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{\cancel{(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}\cdot(3\cdot 2\cdot 1)} = \frac{\overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot\cancel{2}} = 12\cdot 7 = 84$

same

(*b*) $\binom{9}{3} = C(9,3) = \frac{9!}{3!(9-3)!} = \frac{9!}{3!\,6!} = 84$

notice

$C(n,r) = C(n,n-r)$

$\binom{n}{r} = \binom{n}{n-r}$

(*c*) $\binom{9}{8} = C(9,8) = \frac{9!}{8!(9-8)!} = \frac{9!}{8!\,1!} = \frac{9\cdot\cancel{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{\cancel{(8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}\cdot(1)} = 9$

$\binom{9}{8} = \binom{9}{1}$ = number of ways of choosing 1 element from a set of 9.

(*d*) $\binom{9}{9} = C(9,9) = \frac{9!}{9!(9-9)!} = \frac{9!}{9!\,0!} = \frac{9!}{9!\cdot(1)} = \frac{9!}{9!} = 1$

$\binom{9}{9} = \binom{9}{0}$ = number of ways of choosing a subset with no elements.

**[Example 2] (similar to 9.5#12)**

$\overbrace{\{1,2,3,\ldots,1001\}}^{\text{Set } X}$

How many triples of integers chosen from the set {1,2,3, ... ,1001} have a sum that is odd?

"triple" means order is not important, and repetition is not allowed.
In other words, "triples of integers" means "sets of three integers"

Two types: Type A: Triples of 3 odd integers

Type B: Triples of 2 even and 1 odd integer.

We need to know the number of even integers in set $X$

| index | 1 | 2 | 3 | ... | 500 |
|---|---|---|---|---|---|
| integer | $2\cdot1=2$ | $2\cdot2=4$ | $2\cdot3=6$ | ... | $2\cdot500=1000$ |

So there are 500 even integers in set $X$

We also need to know the number of odd integers in set $X$

| index | 0 | 1 | 2 | ... | 500 |
|---|---|---|---|---|---|
| integer | $2\cdot0+1=1$ | $2\cdot1+1=3$ | $2\cdot2+1=5$ | ... | $2\cdot500+1=1001$ |

So there are 501 odd integers in set $X$

Count Set A, the set of triples of odd integers in set X

$$N(A) = C(501,3) = \binom{501}{3} = \frac{501!}{3!(501-3)!} = \frac{501!}{3!\,498!} =$$

$$= \frac{\overset{167}{\cancel{501}}\cdot\overset{250}{\cancel{500}}\cdot 499}{\cancel{3}\cdot\cancel{2}} = 167\cdot 250\cdot 499 = 20{,}833{,}250$$

Count Set B, the set of triples with two even integers and one odd integer.

Task #1 Choose two even integers    Task #2 Choose one odd integer

$$N(B) = C(500,2)\cdot C(501,1)$$

$$= \binom{500}{2}\cdot\binom{501}{1}$$

$$= 62{,}499{,}750$$

So total number of triples is $N(A) + N(B) = 83{,}333{,}000$

**[Example 3] (similar to 9.5#16)**

A jar contains a total of 100 jellybeans. 7 are black licorice flavored, which is a vile flavor and should not even be used as a flavor for candy.

A sample of 10 beans is to be taken from the jar.

**(a)** How many different 10-bean samples are possible?

$$C(100,10) = \frac{100!}{10!\,(100-10)!} = \frac{100!}{10!\,90!} = 17{,}310{,}309{,}456{,}440$$

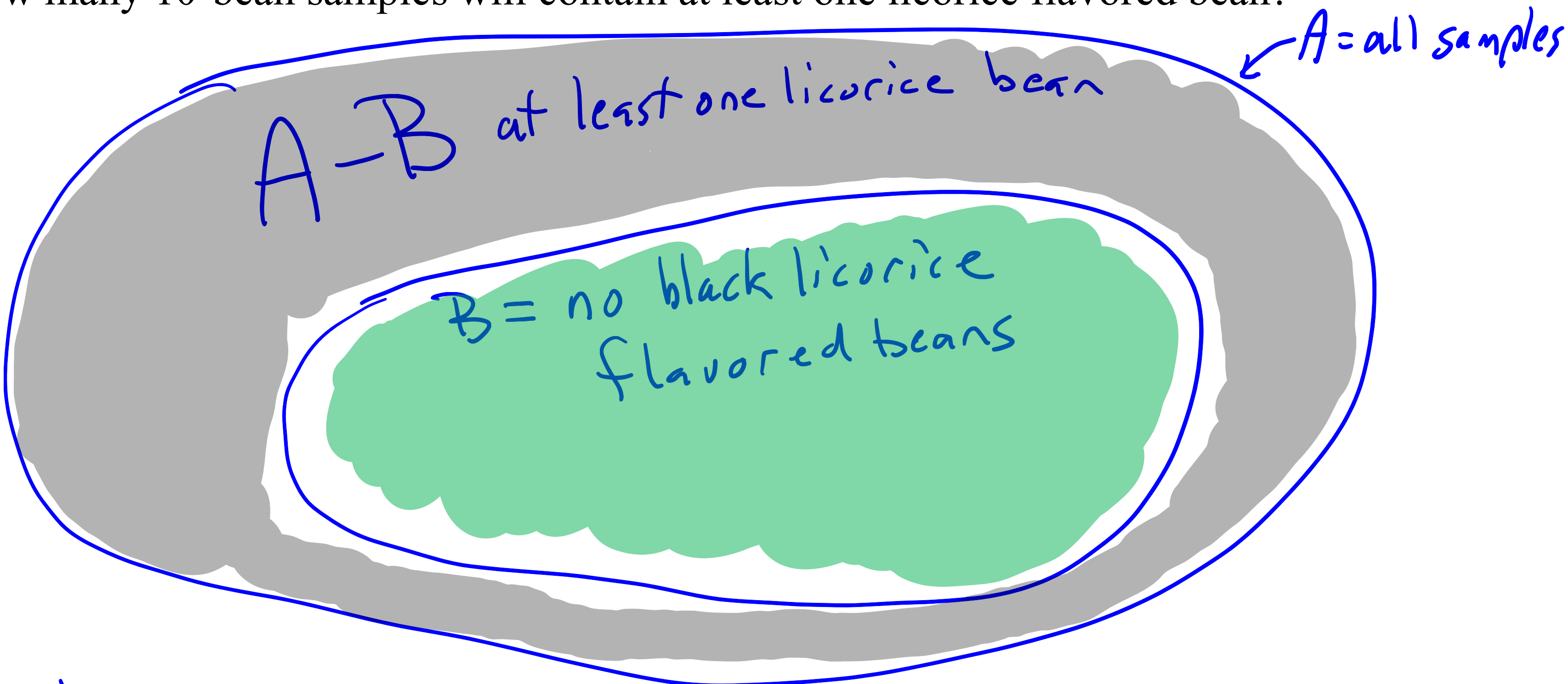
wolfram

**(b)** How many different 10-bean samples are possible that do not contain any licorice flavored beans?

93 beans are not licorice flavored

$C(93,10) = \underset{\text{wolfram}}{\cdots} = 8,079,421,007,658$

**(c)** How many 10-bean samples will contain at least one licorice flavored bean?




$$N(A-B) = N(A) - N(B) = \text{huge number from (a)} - \text{huge number from (b)}$$

$$\overset{\text{wolfram}}{=} C(100,10) - C(93,10)$$

$$= 9{,}230{,}888{,}448{,}782$$

**(d)** What is the probability that a randomly chosen 10-bean sample will contain at least one licorice flavored bean?

$$P(A-B) = \frac{N(A-B)}{N(A)} = \frac{N(A)-N(B)}{N(A)} = \frac{N(A)}{N(A)} - \frac{N(B)}{N(A)} = 1 - \frac{N(B)}{N(A)}$$

$$= 1 - \frac{C(93,10)}{C(100,10)} \overset{\text{wolfram}}{=} \ldots = \frac{6,466,807}{12,126,940} \approx 0.53$$

exact

decimal approximation

**[Example 4] (similar to 9.5#20)**

**(a)** How many distinguishable ways can the letters of the word *MASSACHUSETTS* be arranged in an ordered list?

| | |
|---|---|
| A | 2 |
| C | 1 |
| E | 1 |
| H | 1 |
| M | 1 |
| S | 4 |
| T | 2 |
| U | 1 |
| total | 13 |

$\overline{\ \ }\ \overline{\ \ }\ \overline{\ \ }\ \cdots\ \overline{\ \ }\ \overline{\ \ }$
$1\quad 2\quad 3\quad \cdots\quad 12\quad 13$

Choose spot for As · Choose spots for C · Choose spot for E · Choose spot for H · Choose spot for M · Choose spots for Ss · Choose spots for Ts · Choose spot for U

$$C(13,2)\cdot C(11,1)\cdot C(10,1)\cdot C(9,1)\cdot C(8,1)\cdot C(7,4)\cdot C(3,2)\cdot C(1,1)$$

$$\binom{13}{2}\cdot\binom{11}{1}\cdot\binom{10}{1}\cdot\binom{9}{1}\cdot\binom{8}{1}\cdot\binom{7}{4}\cdot\binom{3}{2}\cdot\binom{1}{1}$$

64,864,800 ways $= \dfrac{13!}{2!1!1!1!1!4!2!1!}$

**The technique that we used in solving part (a) is the essence of Theorem 9.5.2**

**Theorem 9.5.2 Permutations with Sets of Indistinguishable Objects**

Suppose a collection consists of $n$ objects of which

$n_1$ are of type 1 and are indistinguishable from each other

$n_2$ are of type 2 and are indistinguishable from each other

$\vdots$

$n_k$ are of type $k$ and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the $n$ objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!n_3!\cdots n_k!}.$$

**(b)** How many distinguishable orderings begin with an *M* and end with an *S*?

| | ⓐ | ⓑ |
|---|---|---|
| A | 2 | 2 |
| C | 1 | 1 |
| E | 1 | 1 |
| H | 1 | 1 |
| M | 1 | 0 |
| S | 4 | 3 |
| T | 2 | 2 |
| U | 1 | 1 |
| total | 13 | 11 |

$$\underset{1}{M}\ \underset{2}{\_}\ \underset{3}{\_}\ \cdots\ \underset{12}{\_}\ \underset{13}{S}$$

Using Theorem 9.5.2

Number of ways will be

$$\frac{11!}{(2!)(\cancel{1!})(\cancel{1!})(\cancel{1!})(\cancel{0!})(3!)(2!)(\cancel{1!})}$$

$$\frac{11!}{2!\,3!\,2!} = \frac{11!}{2\cdot 6\cdot 2} = \frac{11!}{24} = 1,663,200$$

**(c)** How many distinguishable orderings contain the letters *CH* next to each other in order and also the letters *TT* next to each other in order?

| | (a) | (b) | (c) |
|---|---|---|---|
| A | 2 | 2 | 2 |
| C | 1 | 1 | 0 |
| E | 1 | 1 | 1 |
| H | 1 | 1 | 0 |
| M | 1 | 0 | 1 |
| S | 4 | 3 | 4 |
| T | 2 | 2 | 0 |
| U | 1 | 1 | 1 |
| total | 13 | 11 | 9 |

Tasks

Task #1 Place the CH

Task #2 Place the TT

Task #3 place the 9 other letters.

## Task #1

__ __ __ __ __ __ __ __ __ __ __ __ __
1 2 3 4 5 6 7 8 9 10 11 12 13

Choose location for the C. Then the H will go next to it.

$n_1 = 12$

## Task #2

Choose location for the left T. (Then the other will go next to it)

CH in far left

C H _ _ _ _ _ _ _ _ _ _ X

1 2 3 4 5 6 7 8 9 10 11 12 13

10 spots

CH in far right

_ _ _ _ _ _ _ _ _ _ X C H

1 2 3 4 5 6 7 8 9 10 11 12 13

10 spots

CH not on the end

X C H _ _ _ _ _ _ _ _ _ X

1 2 3 4 5 6 7 8 9 10 11 12 13

9 spots

_ _ _ X C H _ _ _ _ _ _ X

1 2 3 4 5 6 7 8 9 10 11 12 13

9 spots

Set of all orderings = Orderings with C in blank 1 $\cup$ orderings with C in blank 12 $\cup$ orderings with C in blanks 2–11

$$S = A \cup B \cup C$$

Count number of elements in set A

task

#1 choose spot for C — $n_1 = 1$ (far left)

#2 choose spot for H — $n_2 = 1$ (next to C)

#3 Choose spot for left T — $n_3 = 10$ available good spots

#4 Choose spot for right T — $n_4 = 1$ (next to left T)

#5 Choose spots for A, E, M, S, U — $n_5 = \dfrac{9!}{2!\,1!\,1!\,4!\,1!}$

$$N(A) = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot n_5 = 10 \cdot \frac{9!}{2!\,4!} = \frac{10!}{2 \cdot 24} = 75{,}600$$

$N(B)$ will be the Same as $N(A) = 75,600$

Now count # of elements in set C.

task

#1 choose spot for C $\quad n_1 = 10$

#2 Choose spot for H $\quad n_2 = 1$ (next to C)

#3 Choose spot for left T $\quad n_3 = 9$ available ~~good~~ spots

#4 Choose spot for right T $\quad n_4 = 1$ (next to left T)

#5 Choose spots for A, E, M, S, U $\quad n_5 = \dfrac{9!}{2!\,1!\,1!\,4!\,1!}$

$$N(C) = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot n_5 = 10 \cdot 9 \cdot \frac{9!}{2!\,4!} = 9 \cdot \frac{10!}{2 \cdot 24} = 9(75,600)$$

$$= 680400$$

$$\text{So } N(S) = N(A) + N(B) + N(C) = N(A) + N(A) + 9 \cdot N(A) = 11 \cdot N(A) = 11(75,600)$$

$$= 831,600$$