Video for Homework H09.6 Multisets

Reading: Section 9.6 r-Combinations with Repetition Allowed

Homework: H09.6: 9.6#4,6,9,12,14,17

Topics:

- **Counting collections where repetition is allowed**
- **Definition of *multiset***
- **Counting the *number* of *multisets***
- **Counting iterations of a loop**
- **Counting triples of a certain type**
- **Counting r-tuples of a certain type**

We will use the concept of *combinations* from Section 9.5

Definition of *r-combination*

An ***r-combination*** of a set of *n* elements is an *unordered* selection of *r* elements taken from the set of *n* elements. That is, it is a subset of *r* elements.

The *number* of *r-combinations* of a set of *n* elements is denoted $C(n,r)$ or $\binom{n}{r}$. This quantity is spoken "*n choose r*".

Theorem 9.5.1 Computational Formula for $\binom{n}{r}$

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

[Example 1] (Similar to 9.6#18)

A large can of coins consists of pennies, nickels, dimes, and quarters.

(Coins of a particular denomination are indistinguishable from one another.)

(a) Suppose the collection contains at least 20 coins of each type.

How many different collections of 20 coins can be chosen?

In order to solve this problem, it is helpful to introduce some notation and to discuss various ways of notating collections of coins.

Introduce letters to denote the types of coins.

Let P denote *Penny*.

Let K denote *Nickel*.

Let D denote *Dime*.

Let Q denote *Quarter*.

And let X be the set of all types of coins. That is, $X = \{P, K, D, Q\}$

Let n be the number of elements in set X. So $n = 4$

Let r be the number of coins in the collection that we are going to chose. So $r = 20$.

One such collection could be made up of the following twenty coins:

$$\underbrace{P,P,P,P,P,P}_{6\,P},\underbrace{K,K,K,K,K,K,K,K,K}_{9\,K},\underbrace{D,D,D}_{3\,D}\,\underbrace{Q,Q}_{2\,Q}$$

The terminology of *multiset* applies to this kind of collection

Definition of *multiset*

words: a *multiset of size* r *chosen from a set* X

alternate words: an r *combination with repetition allowed, chosen from a set* X

meaning: an unordered selection of r elements taken from set X with repetition allowed

symbol: $[x_1, x_2, \dots, x_r]$, where each $x_k \in X$ and some x_k may equal each other.

With this notation, the collection of twenty coins from the previous page would be denoted

$$[P,P,P,P,P,P,K,K,K,K,K,K,K,K,K,D,D,D,Q,Q]$$

And with this terminology, we can articulate the question that we have been asked:

(a) How many *multisets* of size $r = 20$ can be chosen from a set X that has $n = 4$ elements?

We have to figure out a way to count the number of such multisets.

It turns out that if we use a simpler method of displaying a particular collection, the counting is quite easy. We will work our way to a simple way of presenting collections in the next few pages.

First, suppose there are 20 boxes to be filled with letters chosen from the set $X = \{P, K, D, Q\}$. The sample collection discussed above would be displayed as a table.

P	P	P	P	P	P	K	K	K	K	K	K	K	K	K	D	D	D	Q	Q
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Note the number 20 is the value of r. In general, the number of boxes needed to display a multiset in this manner would be

$$number\ of\ boxes = r$$

Now, instead, imagine that there are vertical bars, I, put in to separate the different types of letters P, K, D, Q, and then simple x symbols to denote the letters filling the blanks. The sample collection presented above would then be displayed in the following way.

x	x	x	x	x	x	I	x	x	x	x	x	x	x	x	x	I	x	x	x	I	x	x
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

Note that, of course, three more boxes are needed to make room for the three vertical bars that are used to separate the different kinds of letters. And note that

$$3 = 4 - 1 = n - 1$$

In general, the number of vertical bars needed to separate the different kinds of elements drawn from a set X with n elements would be

$$number\ of\ vertical\ bars = n - 1$$

The total number of boxes is now 23. Note that

$$23 = 20 + 3 = 20 + (4 - 1) = r + n - 1$$

So, in general, the total of number of boxes needed to display a multiset in this way is

$$number\ of\ boxes = r + n - 1$$

Of course, the row of numbers across the bottom are not really necessary. The collection above could be displayed more consisely as

x	x	x	x	x	x	I	x	x	x	x	x	x	x	x	x	I	x	x	x	I	x	x

In fact, we could leave out the x symbols, and just show the vertical bars in their cells:

						I										I				I		

If the vertical bars are placed in adjacent cells, that would indicate that certain elements of set X are not included in the collection. For instance, the following table

x	x	x	x	x	x	I	x	x	x	x	x	x	x	x	x	x	x	x	I	I	x	x

or, more concisely,

						I													I	I		

Would represent this collection

$$\underbrace{P,P,P,P,P,P}_{6\,P},\underbrace{K,K,K,K,K,K,K,K,K,K,K,K}_{12\,K},\underbrace{Q,Q}_{2\,Q}$$

And the vertical bars can appear at the end of the row of boxes. For instance, the table

x	x	x	x	x	x	I	x	x	x	x	x	x	x	x	x	x	x	x	x	x	I	I

or, more concisely,

						I															I	I

would represent this collection

$$\underbrace{P,P,P,P,P,P}_{6\,P},\underbrace{K,K,K,K,K,K,K,K,K,K,K,K,K,K}_{14\,K}$$

We see that every choice of 3 of the 23 cells for the vertical bars corresponds to a particular multiset of 20 coins chosen from set ~~K~~ X = $\{P, K, D, Q\}$.

That is, every choice of a subset of 3 cells chosen from the set of 23 cells corresponds to a particular multiset of 20 coins chosen from set $K = \{P, K, D, Q\}$.

Therefore, the ~~number of~~ number of multisets will be the number of 3-combinations of a set of 23 elements.

$$\binom{23}{3}$$

This number is easy to compute.

$$C(23,3) = \binom{23}{3} = \frac{23!}{3!(23-3)!} = \frac{23!}{3!\,20!} = \frac{23 \cdot 22 \cdot 21}{3 \cdot 2} = 23 \cdot 11 \cdot 7 = \cdots = 1771$$

We have found the answer to question (a). That is, this is the number of different collections of 20 coins that can be chosen. is 1771

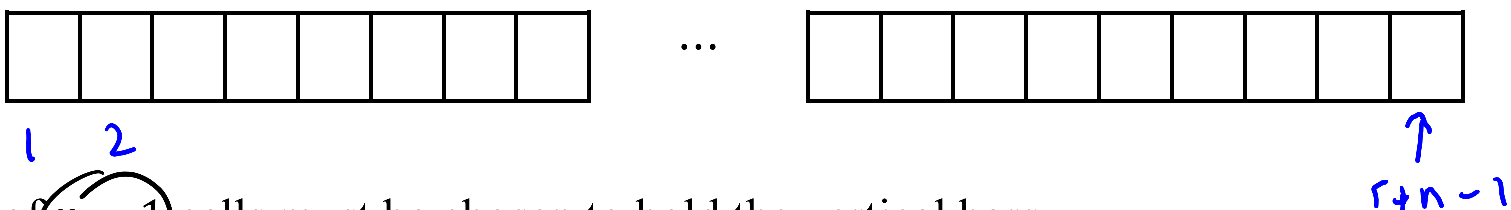
Now, we can generalize the counting technique. Here is our goal.

Goal: Count the number of multisets of size r that can be selected from a set X of n elements.

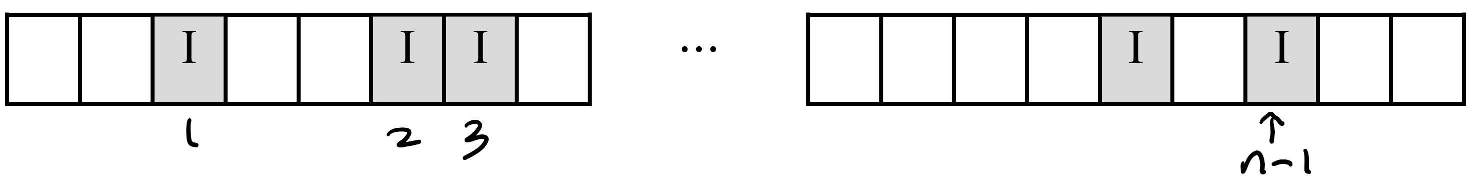
Imagine an empty row of cells in a table.

- There will need to be r cells to hold the elements chosen to be in the multiset.
- There will need to be $n-1$ cells to hold the vertical bars, I, that separate the various types of elements in the multiset.

So, the table will need $r+n-1$ cells.



A subset of $n-1$ cells must be chosen to hold the vertical bars.



So, the number of multisets will be the number of subsets of $n-1$ cells chosen from the row of $r+n-1$ cells.

In other words, the number of multisets will be the number of $(n-1)$-combinations chosen from a set of $(r+n-1)$ elements

$$C(r+n-1, n-1) = \binom{r+n-1}{n-1}$$

Compare the result that we just reached to the following theorem from the book.

Theorem 9.6.1

The number of r-combinations with repetition allowed (or multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Notice that the book's theorem presents a formula that looks different from our formula.

But in fact, the two formulas are the same, as we can see if we write the factorial expressions that correspond to the two formulas.

The formula that we reached:

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!\,\big((r+n-1)-(n-1)\big)!} = \frac{(r+n-1)!}{(n-1)!\,r!}$$

The formula in the book's Theorem 9.6.1:

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!\,\big((r+n-1)-r\big)!} = \frac{(r+n-1)!}{r!\,(n-1)!}$$

these match

Because the two formulas are the same, and because there is potential confusion, it is worthwhile to give a more complete presentation of Theorem 9.6.1

Theorem 9.6.1

The number of *multisets* of size r that can be chosen from a set of n elements is

$$C(r+n-1, n-1) = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!\,r!} = \binom{r+n-1}{r} = C(r+n-1, r)$$

(b) Suppose the collection contains at least 20 coins of each type.

How many different collections of 20 coins can be chosen that contain at least 15 pennies?

Pick out 15 pennies to go in the collection.

Now the task is to choose a multiset of 5 coins for the rest of the collection.

$r = 5$ size of the multiset

$n = 4$ number of elements in set X (the number of types of coins)

So the number of multisets is

$$\binom{r+n-1}{r} = \binom{5+4-1}{5} = \binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 8 \cdot 7 = \boxed{56}$$

(c) Suppose the collection contains at least 20 coins of each type.

How many different collections of 20 coins can be chosen that contain at most ~~15~~ 14 pennies?

Green Set = Blue Set − Red Set

N(Green) = N(Blue) − N(Red)

= 1771 − 56

= 1715 = number of collections containing at most 14 pennies

(d) Suppose the collection contains only ~~15~~ 14 pennies, but least 20 coins of each other type. How many different collections of 20 coins can be chosen?

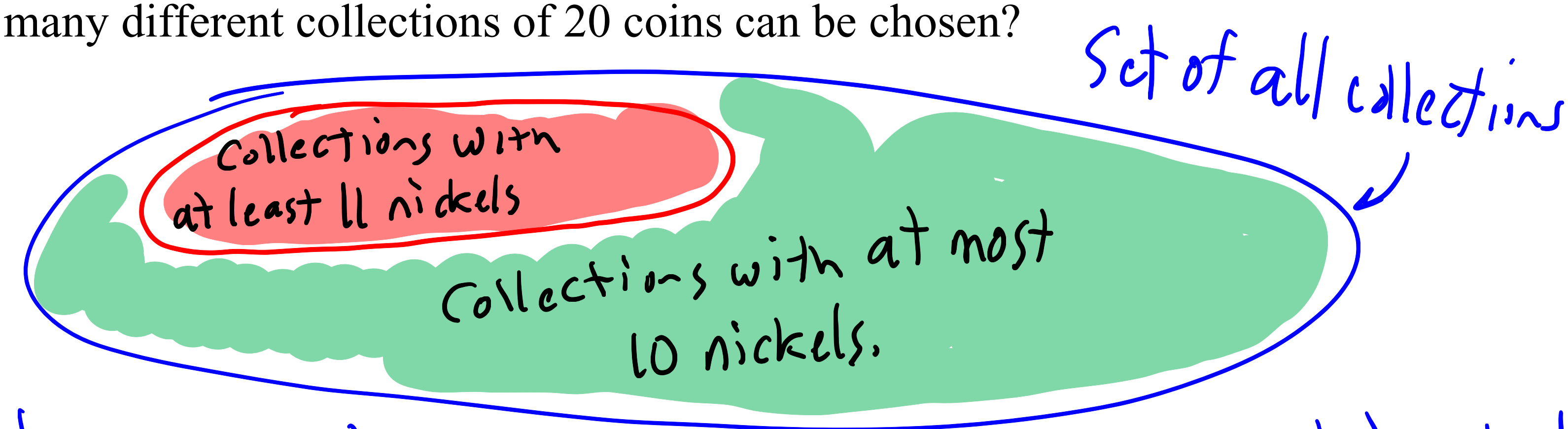
This is the wording of your similar homework problem from the book.

This is another way of asking the same question that was asked in (c).

The answer is 1715

(e) Suppose the collection contains only 10 nickels, but least 20 coins of each other type.

How many different collections of 20 coins can be chosen?



Start by counting the number of collections with at least 11 nickels,
Put 11 nickels in collection to begin the collection.
Then choose a multiset of $r=9$ coins from the set X that has $n=4$ elements.

The number of such multisets is

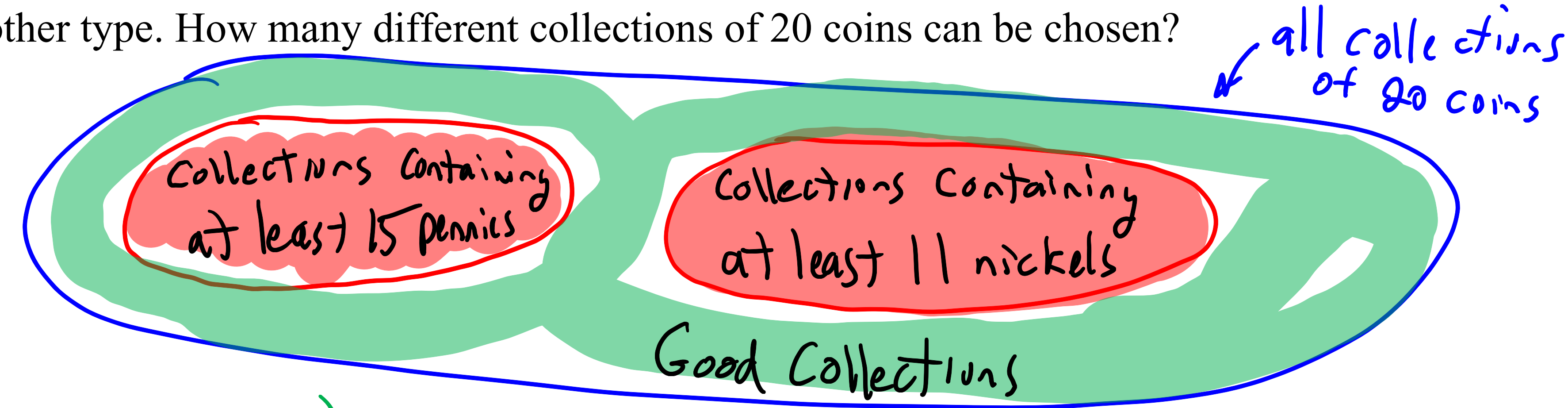
$$\binom{r+n-1}{r} = \binom{9+4-1}{9} = \binom{12}{9} = \frac{12!}{9!(12-9)!} = \frac{12!}{9!3!} = \frac{12\cdot 11\cdot 10}{3\cdot 2} = 2\cdot 11\cdot 10 = 220$$

N(collections with at most 10) = N(collections) − N(collections with at least 11)

$= 1771 - 220$

$= 1551$

(f) Suppose the collection contains only ~~15~~ 14 pennies and 10 nickels, but least 20 coins of each other type. How many different collections of 20 coins can be chosen?



$$N(\text{Good Collections}) = N(\text{All collections}) - N(\text{Collections with} \geq 15 \text{ pennies}) - N(\text{Collections with} \geq 11 \text{ nickels})$$

$$= 1771 - 56 - 220$$

$$= 1695$$

End of [Example 1]

[Example 2] (Similar to 9.6#9) Counting iterations of a loop.

Consider the following algorithm segment

```
for k := 1 to 10
    for j := k to 10
        for i := j to 10
            [Statements in body of inner loop.
            None contains branching statements that lead outside the loop.]
        next i
    next j
next k
```

How many times will the innermost loop be iterated when the algorithm segment is run?

The key to this problem is to realize that the inner loop will be iterated once for each triple of integers (i, j, k) such that $10 \geq i \geq j \geq k \geq 1$

Examples of such triples are (6,5,3) or (9,1,1), or (10,3,2), etc.

So, our question has become the following:

How many triples of integers (i, j, k) are there such that ? $10 \geq i \geq j \geq k \geq 1$

It turns out that if we use a visual method of displaying a particular triple, the counting is quite easy. We will work our way to a simple way of presenting ~~collections~~ triples in the next few pages.

Consider ten cells holding the integers from 1 to 10.

1	2	3	4	5	6	7	8	9	10

Now consider those ten cells being separated by nine cells holding the letter I, to signify dividers separating the integers from 1 to 10

1	I	2	I	3	I	4	I	5	I	6	I	7	I	8	I	9	I	10

Now imagine the white cells being empty, but still being the spots where the integers from 1 to 10 would go.

	I		I		I		I		I		I		I		I		I	

Now consider inserting letters into those empty white cells to indicate the value of a variable.

For example, the table

	I		I		I		I		I		I		I	k	I		I	

indicates that $k = 8$

And the table

k	I		I		I		I		I		I		I		I		I	

indicates that $k = 1$

We can insert three letters into the empty white cells to indicate the value of a triple.

For example, the table

	I	k	I		I		I	j	I		I		I	i	I		I	

represents the triple $(i, j, k) = (8{,}5{,}2)$

and the table

	I	kj	I		I		I		I		I		I		I		I	i

represents the triple $(i, j, k) = (10{,}2{,}2)$

Realize that because the integers i, j, k must satisfy $i \geq j \geq k$, we don't really need the letters, themselves. We can just use a placeholder like the symbol x.

For example, the table

	I	x	I		I		I	x	I		I		I	x	I		I	

represents the triple $(i, j, k) = (8,5,2)$

and the table

	I	xx	I		I		I		I		I		I		I		I	x

represents the triple $(i, j, k) = (10,2,2)$

Now, realize that we don't need to show the empty white cells. We really only need to show the positions of the three x symbols in relation to the nine I symbols.

For example, the table

I	x	I	I	I	x	I	I	I	x	I	I

represents the triple $(i, j, k) = (8,5,2)$

and the table

I	x	x	I	I	I	I	I	I	I	I	x

represents the triple $(i, j, k) = (10,2,2)$

We see that the table representing a triple (i, j, k) will always have 12 cells: Three cells to hold x symbols and nine cells to hold I symbols.

We see that every choice of 3 of the 12 cells for the x symbols corresponds to a particular triple of integers (i, j, k) such that $10 \geq i \geq j \geq k \geq 1$

That is, every choice of a subset of 3 cells chosen from the set of 12 cells corresponds to a particular triple of integers (i, j, k) such that $10 \geq i \geq j \geq k \geq 1$

Therefore, the number of number of triples will be the number of 3-combinations of a set of 10 elements.

$$\binom{10}{3}$$

This number is easy to compute.

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 10 \cdot 3 \cdot 4 = 120$$

We have found the answer to our original question!

The number of times that the inner loop will be iterated is equal to the number of triples of integers (i, j, k) such that $10 \geq i \geq j \geq k \geq 1$, which is 120

End of [Example 2]

Generalize this result to counting r-*tuples* of a certain type

We can generalize the counting technique. Here is our goal.

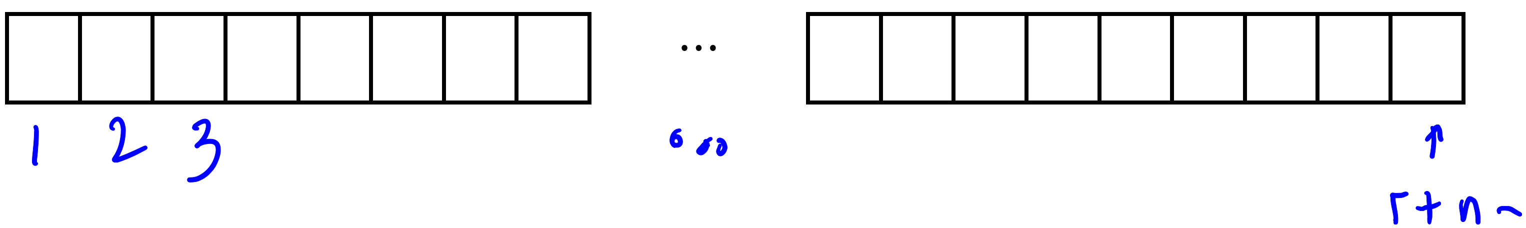
Goal: Count the number of r-*tuples* of integers $(m_1, m_2, \dots, m_r)$ such that

$$n \geq m_1 \geq m_2 \geq \cdots \geq m_r \geq 1$$

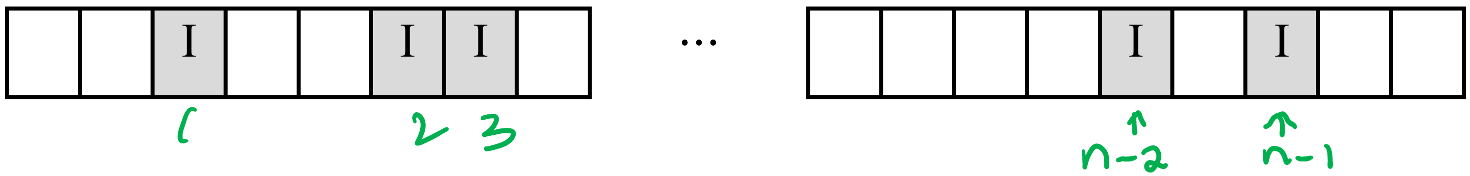
Imagine an empty row of cells in a table.

- There will need to be r cells to hold the x symbols denoting the integers $m_1, m_2, \dots, m_r$.
- There will need to be $n - 1$ cells to hold the vertical bars, I, that separate the spots representing possible integer values from 1 to n.

So, the table will need $r + n - 1$ cells.



A subset of $n-1$ cells must be chosen to hold the vertical bars.

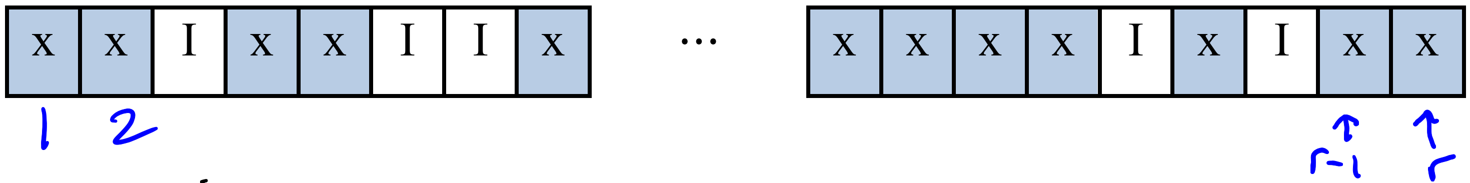


So, the number of ~~multisets~~ r-tuples will be the number of subsets of $n-1$ cells chosen from the row of $r+n-1$ cells.

In other words, the number of ~~multisets~~ r-tuples will be the number of $(n-1)$-combinations chosen from a set of $(r+n-1)$ elements

$$C(r+n-1, n-1) = \binom{r+n-1}{n-1}$$

Equivalently, a subset of r cells must be chosen to hold the x symbols.



So, the number of ~~multisets~~ r-tuples will be the number of subsets of r cells chosen from the row of $r + n - 1$ cells.

In other words, the number of ~~multisets~~ r tuples will be the number of r-combinations chosen from a set of $(r + n - 1)$ elements

$$C(r+n-1, r) = \binom{r+n-1}{n-1}$$

These two approaches yield the same result:

Theorem about the number of a certain type of *r-tuple*

The number of r-*tuples* of integers $(m_1, m_2, \dots, m_r)$ such that $n \geq m_1 \geq \cdots \geq m_r \geq 1$ is

$$C(r+n-1, n-1) = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!\,r!} = \binom{r+n-1}{r} = C(r+n-1, r)$$