Video for Homework H09.7 Pascal's Formula and the Binomial Theorem

Reading: Section 9.7 Pascal's Formula and the Binomial Theorem

Homework: H09.7: 9.7# 7,11,30,32,39,44,46,50

Topics:

- **Particular important values of $C(n, r)$**
- **Pascal's Formula**
- **Pascal's Triangle**
- **The Binomial Theorem**
- **Using the Binomial Theorem**

We will use the concept of *combinations* from Section 9.5

Definition of *r-combination*

An ***r-combination*** of a set of *n* elements is an *unordered* selection of *r* elements taken from the set of *n* elements. That is, it is a subset of *r* elements.

The *number* of *r-combinations* of a set of *n* elements is denoted $C(n, r)$ or $\binom{n}{r}$. This quantity is spoken "*n choose r*".

Theorem 9.5.1 Computational Formula for $\binom{n}{r}$

The number of subsets of size *r* (or *r*-combinations) that can be chosen from a set of *n* elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where *n* and *r* are nonnegative integers with $r \leq n$.

And we will frequently rewrite factorial expressions in different ways

Recall the definition of factorial

$$n! = n \cdot (n-1)(n-2) \cdots (2)(1)!$$

Observe that one factorial expression can be written many ways.

$$\begin{aligned} n! &= n \cdot (n-1)(n-2) \cdots (2)(1) \\ &= n \cdot (n-1)! \\ &= n \cdot (n-1) \cdot (n-2)! \end{aligned}$$

And realize that there are many variations on this idea.

$$(n+1) \cdot n! = (n+1)!$$

\[Example 1\] Particular important values of $\boldsymbol{C(n,r)}$

$$(\boldsymbol{a})\ \ C(n,n) = \binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = \frac{n!}{n!\cdot 1} = 1$$

$$(\boldsymbol{b})\ \ C(n,n-1) = \binom{n}{n-1} = \frac{n!}{(n-1)!\,\big(n-(n-1)\big)!} = \frac{n\cdot(n-1)!}{(n-1)!\,1!} = n$$

$$(\boldsymbol{c})\ \ C(n,n-2) = \binom{n}{n-2} = \frac{n!}{(n-2)!\,\big(n-(n-2)\big)!} = \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!\,2!} = \frac{n\cdot(n-1)}{2}$$

$$(\boldsymbol{d})\ \ C(n,0) = \binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{0!\,n!} = \frac{n!}{1\cdot n!} = 1$$

$$(\boldsymbol{e})\ \ C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{\big(n-(n-r)\big)!\,(n-r)!} = \binom{n}{n-r} = C(n,n-r)$$

trick

End of \[Example 1\]

Pascal's Formula

Theorem 9.7.1 Pascal's Formula

For all $n, r \in Z$ with $1 \leq r \leq n$, $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Algebraic Proof of Pascal's Formula

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!\big(n-(r-1)\big)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r+1)!} \cdot \left(\frac{r}{r}\right) + \frac{n!}{r!\,(n-r)!} \cdot \left(\frac{n-r+1}{n-r+1}\right)$$

$$= \frac{n! \cdot r}{r \cdot (r-1)!\,(n-r+1)!} + \frac{n! \cdot (n+1-r)}{r!\,(n-r+1) \cdot (n-r)!}$$

$$= \frac{n! \cdot r}{r!\,(n-r+1)!} + \frac{n! \cdot (n+1) - n! \cdot (r)}{r! \cdot (n-r+1)!}$$

$$= \frac{(n+1)!}{r! \cdot (n-r+1)!}$$

$$= \frac{(n+1)!}{r! \cdot \big((n+1)-r\big)!}$$

$$= \binom{n+1}{r}$$

Combinatorial Proof of Pascal's Formula

Combinatorics is an area of math having to do with the counting of sets. A combinatorial proof is one in which we prove the value of a certain quantity by making observations about counting sets.

We can prove Pascal's Formula,

$$\text{For all } n, r \in Z \text{ with } 1 \leq r \leq n, \quad \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

using a combinatorial argument.

Consider the symbol $\binom{n+1}{r}$ on the left side of Pascal's Formula. That symbol represents the number of subsets of r elements that can be chosen from a set that has $n+1$ elements.

Suppose $X = \{x_1, x_2, \dots, x_n, x_{n+1},\}$ is a set containing $n+1$ elements.

Let $\mathcal{X}_r$ be the collection of all *subsets* of X that have r elements. That is,

$$\mathcal{X}_r = \{S \subseteq X \text{ such that } N(S) = r\}$$

Observe that the collection $\mathcal{X}_r$ is a subset of the *set of all subsets* of X. That is, $\mathcal{X}_r \subseteq \mathcal{P}(X)$, the *power set* of X.

Also observe that symbols $C(n+1, r)$ and $\binom{n+1}{r}$ represent the number of sets in $\mathcal{X}_r$.

$$N(\mathcal{X}_r) = C(n+1, r) = \binom{n+1}{r}$$

Next, observe that $\mathcal{X}_r$ can be described as the union of two disjoint subsets

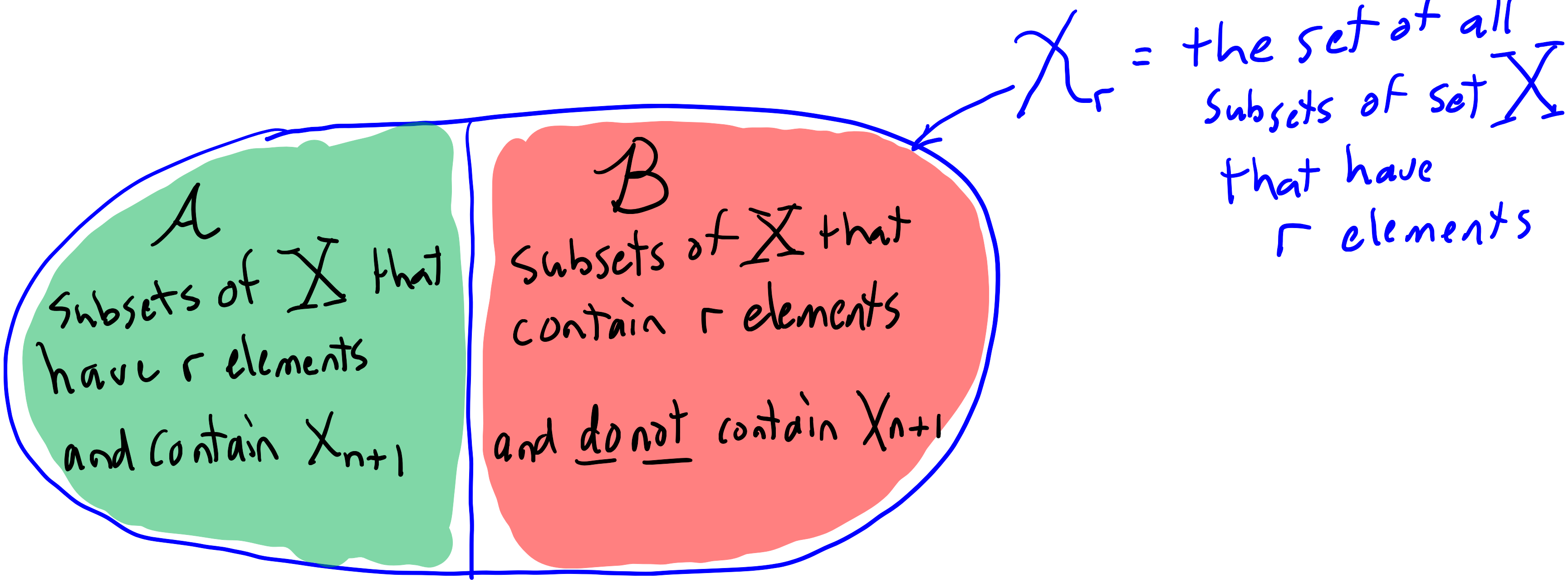
Define the set $\mathcal{A} \subseteq \mathcal{X}_r$ as follows

$$\mathcal{A} = \{A \subseteq X \text{ such that } N(A) = r \text{ and } x_{n+1} \in A\}$$

Define the set $\mathcal{B} \subseteq \mathcal{X}_r$ as follows

$$\mathcal{B} = \{B \subseteq X \text{ such that } N(B) = r \text{ and } x_{n+1} \notin B\}$$

Observe that $\mathcal{A}$ and $\mathcal{B}$ are disjoint, and that $\mathcal{X}_r = \mathcal{A} \cup \mathcal{B}$.



Therefore, by the addition rule,

$$N(\mathcal{X}_r) = N(\mathcal{A}) + N(\mathcal{B})$$

That is,

$$\binom{n+1}{r} = N(\mathcal{A}) + N(\mathcal{B})$$

Finding the values of $N(\mathcal{A})$ and $N(\mathcal{B})$ is fairly simple, so this formula is really useful.

To find $N(\mathcal{A})$, recall that

$$\mathcal{A} = \{A \subseteq X \text{ such that } N(A) = r \text{ and } x_{n+1} \in A\}$$

The number of elements of $\mathcal{A}$ is the number of r element subsets $A \subseteq X$ such that $x_{n+1} \in A$

To count the number of such subsets, consider the task of choosing such a subset as two tasks

Task #1: Choose element x_{n+1} to go in set A

The number of ways to do task #1 is $k_1 = 1$

Task #2: Choose the remaining $r - 1$ elements for set A from the set $\{x_1, x_2, \dots, x_n\}$

The number of ways to do task #2 is

$$k_2 = \binom{n}{r-1}$$

So, the number of ways to choose a subset A is

$$k = k_1 \cdot k_2 = 1 \cdot \binom{n}{r-1}$$

Therefore,

$$N(\mathcal{A}) = \binom{n}{r-1}$$

To find $\boldsymbol{N(\mathcal{B})}$, recall that

$$\mathcal{B} = \{B \subseteq X \text{ such that } N(B) = r \text{ and } x_{n+1} \notin B\}$$

Observe that if $B \subseteq X$ and $x_{n+1} \notin B$, then $B \subseteq \{x_1, x_2, \ldots, x_n\}$

So the number of sets in $\mathcal{B}$ will be the number of r element subsets of a set with n elements.

That is,

$$N(\mathcal{B}) = \binom{n}{r}$$

Substitute expressions for $N(\mathcal{A})$ and $N(\mathcal{B})$ into earlier equation involving $N(\mathcal{X}_r)$

Our earlier equation

$$N(\mathcal{X}_r) = N(\mathcal{A}) + N(\mathcal{B})$$

becomes

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

which is Pascal's Formula

End of Combinatorial Proof of Pascal's Formula

Pascal's Formula can be visualized in Pascal's Triangle

value of b (columns); value of a (rows)

$\binom{a}{b}$	0	1	2	3	4	…	$r-1$	r	…
0	1								
1	1								
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
⋮	⋮	⋮	⋮	⋮	⋮	…	⋮	⋮	
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	…	$\binom{n}{r-1}$	$\binom{n}{r}$	…
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	…	$\binom{n+1}{r-1}$	$\binom{n+1}{r}$	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$$

The Binomial Theorem

A sum of two terms, such as $a + b$ is called a ***binomial***. The binomial theorem gives an expression for nonnegative integer powers of a binomial, $(a + b)^n$.

Theorem 9.7.2 ***Binomial Theorem***

For all real numbers a, b and all nonnegative integers n,

$$(a+b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

Pascal's Formula can be used in the proof of the *Binomial Theorem*.

(There is also a simpler combinatorial proof. See the book for both proofs.)

[Example 2] Using the Binomial Theorem

(a) Find the expansion of $(a + b)^4$ by using the *Binomial Theorem*.

$(a+b)^4 \leftarrow n=4$

binomial theorem with $n=4$

$$= \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k$$

$$= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4$$

$$= 1 \cdot a^4 \cdot 1 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1 \cdot 1 \cdot b^4$$

$$= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 2 \cdot 3 = 6$$

(b) Find the coefficient of x^6 when $(2x+3)^{10}$ is expanded by the *Binomial Theorem*.

$n=10$

$$(2x+3)^{10} = \sum_{k=0}^{k=10} \binom{10}{k}(2x)^{10-k}\cdot(3)^k$$

Binomial Theorem with $n=10$

The power x^6 will appear in the term that has $k=4$.

That Term will be

$$\binom{10}{4}(2x)^{10-4}3^4 = \frac{10!}{4!(10-4)!}\cdot(2x)^6\cdot 3^4 = \frac{10!}{4!\,6!}\cdot 2^6\cdot x^6\cdot 3^4$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{(4\cdot 3\cdot 2\cdot 1)\cdot 6!}\cdot 64\cdot 81\cdot x^6 = 10\cdot 3\cdot 7\cdot 64\cdot 81\cdot x^6$$

$$= 1{,}088{,}640\,x^6$$

coefficient is 1,088,640

(c) Find the coefficient of $u^8 v^{10}$ when $(u^2 - v^2)^9$ is expanded by the *Binomial Theorem*.

$$(u^2 - v^2)^9 = \sum_{k=0}^{k=9} \binom{9}{k} (u^2)^{9-k} (-v^2)^k$$

$n=9$

binomial theorem with $n=9$

The expression $u^8 v^{10}$ will show up in the $k=5$ term.

Build the $k=5$ term

$$\binom{9}{5}(u^2)^{9-5}(-v^2)^5 = \binom{9}{5}(u^2)^4(-v^{10}) = -\binom{9}{5} u^8 v^{10}$$

Coefficient is $-\binom{9}{5} = -126$

End of [Example 2]

Recall the definition of a *Closed Form Expression* from the video for Homework H05.2

Definition of Closed Form Expression

A ***closed form expression*** is a mathematical expression that involves a known (finite) number of standard operations.

[Example 3] Use the *Binomial Theorem* to simplify the sums, writing each in *closed form* without a summation symbol.

$$(\boldsymbol{a}) \quad \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} 13^{n-k} = \sum_{k=0}^{k=n} \binom{n}{k} 13^{n-k} (-1)^k$$

$a = 13$, $b = -1$

Binomial theorem

$$= (13 + (-1))^n$$

$$= 12^n$$

(b) $\sum_{k=0}^{k=n} \binom{n}{k} 13^k$ $= \sum_{k=0}^{k=n} \binom{n}{k} (1)^{n-k} 13^k$

trick

this is just the number 1

$a = 1$

$b = 13$

use binomial theorem

$= (1 + 13)^n$

$= \boxed{14^n}$

(c) $\sum_{k=0}^{k=n} \binom{n}{k} \frac{1}{13^k} = \sum_{k=0}^{k=n} \binom{n}{k} \left(\frac{1}{13}\right)^k = \sum_{k=0}^{k=n} \binom{n}{k} (1)^{n-k} \left(\frac{1}{13}\right)^k$

$a = 1$ $\quad b = \frac{1}{13}$

use binomial theorem

$= \left(1 + \frac{1}{13}\right)^n$

$= \left(\frac{14}{13}\right)^n$

(**d**) $\sum_{k=0}^{k=n} \binom{n}{k} x^k 13^{n-k}$ $= \sum_{k=0}^{k=n} \binom{n}{k} 13^{n-k} x^k$

$a = 13$ $b = x$

use binomial theorem

$= (13 + x)^n$

$= (13+x)^n$

End of [Example 3]

End of Video for Homework H09.7