Video for Homework H10.1a Trails, Paths, Walks, Circuits

Reading: From Chapter 10 Theory of Graphs and Trees

- Section 10.1 Trails, Paths, Circuits
  - pages 677 – 683, Examples 10.1.1 – 10.1.5

Homework: H10.1a: 10.1 # 2,3,5,8

Topics:

- **Review of Basic Graph Definitions from Chapter 1**
- **Definitions of Walk, Trail, Path, Circuit**
- **Connectedness**

Recall the definition of *graph* from Section 1.4

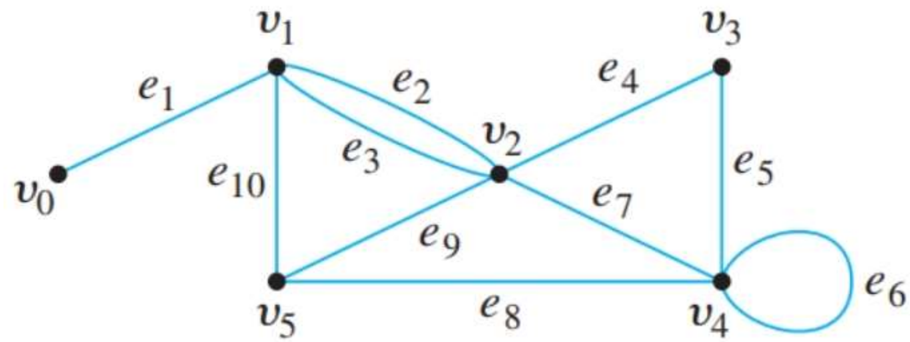
Definition of **Graph**

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

[Example 1] In the graph G shown.



(a) Find $V(G)$

$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$

(b) Find $E(G)$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

(c) Find the edge-endpoint function.

edge	endpoints
e_1	$\{v_0, v_1\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_1, v_2\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_3, v_4\}$
e_6	$\{v_4\}$
e_7	$\{v_2, v_4\}$
e_8	$\{v_4, v_5\}$
e_9	$\{v_2, v_5\}$
e_{10}	$\{v_1, v_5\}$

Section 10.1 begins with the introduction of some new terminology

Definition of **Walk, Trail, Path, Closed Walk, Circuit, Simple Circuit**

Let G be a graph, and let v and w be vertices in G.

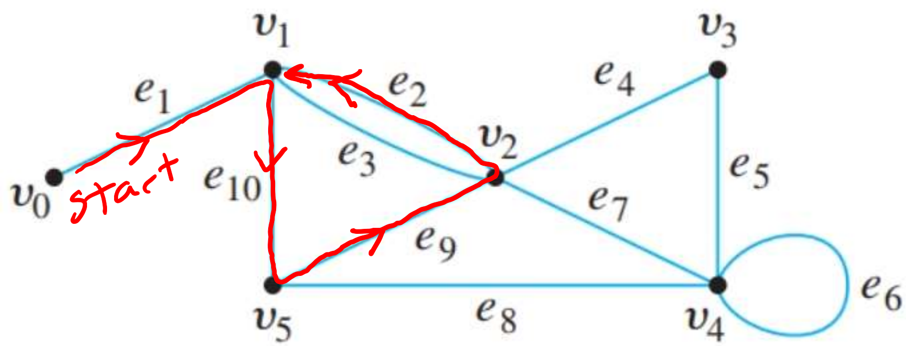
- A **walk from $\boldsymbol{v}$ to $\boldsymbol{w}$** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form $v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$, where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1,2, \ldots, n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from $\boldsymbol{v}$ to $\boldsymbol{v}$** consists of the single vertex v.
- A **trail from $\boldsymbol{v}$ to $\boldsymbol{w}$** is a walk from v to w that does not contain a repeated edge.
- A **path from $\boldsymbol{v}$ to $\boldsymbol{w}$** is a trail that does not contain a repeated vertex.
- A **closed walk** is a walk that starts and ends at the same vertex.
- A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.
- A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

That definition is immediately followed by a nice table:

Summary of Definitions of **Walk, Trail, Path, Closed Walk, Circuit, Simple Circuit**

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed Walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	yes	yes

[Example 2] Return to the graph G from **[Example 1]**.



Classify the following walks.

That is, decide if each is a trail, path, closed walk, circuit, simple circuit, or just a walk.

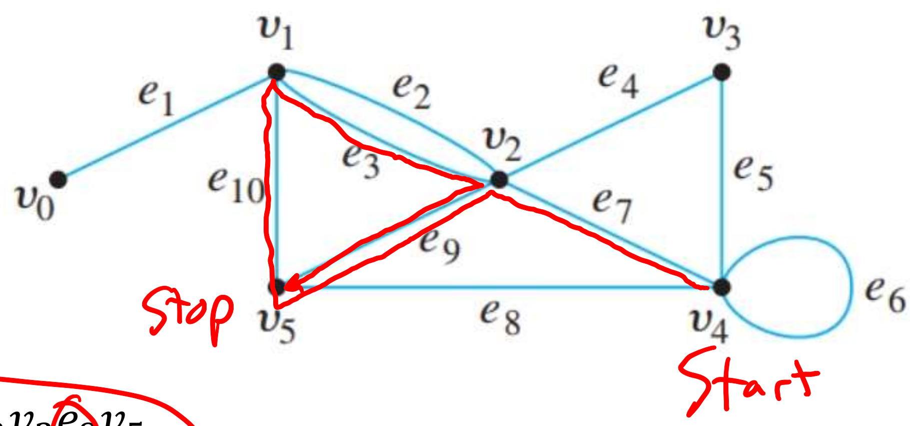
(a) $v_0 e_1 v_1 e_{10} v_5 e_9 v_2 e_2 v_1$

walk ✓

trail ✓ (no repeated edges)

(not a path (has a repeated vertex))

(not a closed walk (does not begin + end at same point))

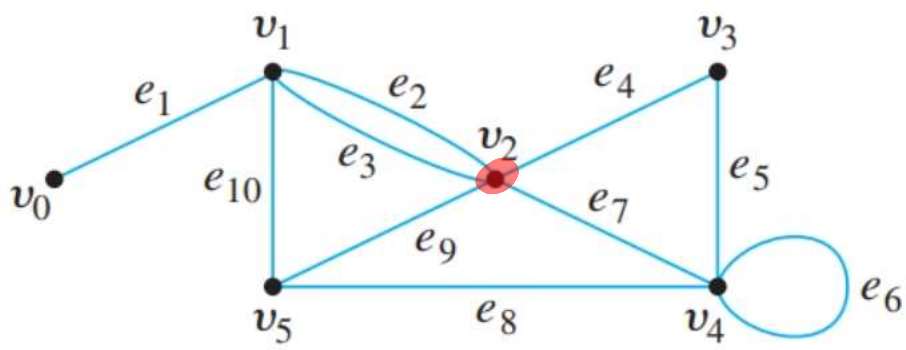


(b) $v_4 e_7 v_2 e_9 v_5 e_{10} v_1 e_3 v_2 e_9 v_5$

has repeated edge

does not start + stop at same vertex

Just a walk



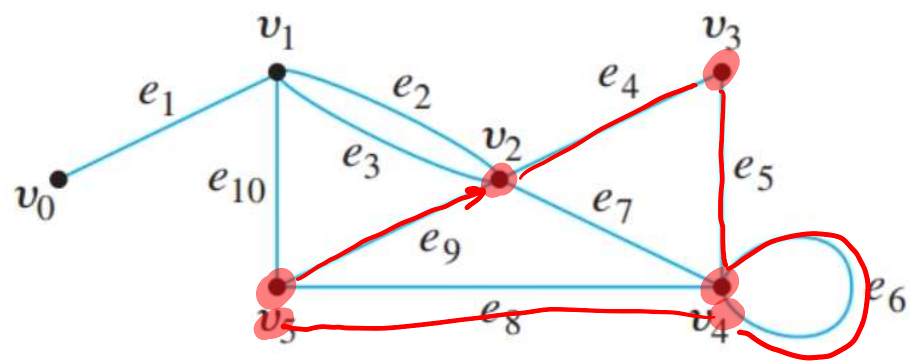
(c) v_2

Walk ✓

Trail ✓

~~Path no~~ starts & stops at same vertex

Mistake in video!
Should also have said Closed Walk!



(d) $v_5 v_2 v_3 v_4 v_4 v_5$

$v_5 e_9 v_2 e_4 v_3 e_5 v_4 e_6 v_4 e_8 v_5$

<u>Circuit</u>

walk ✓

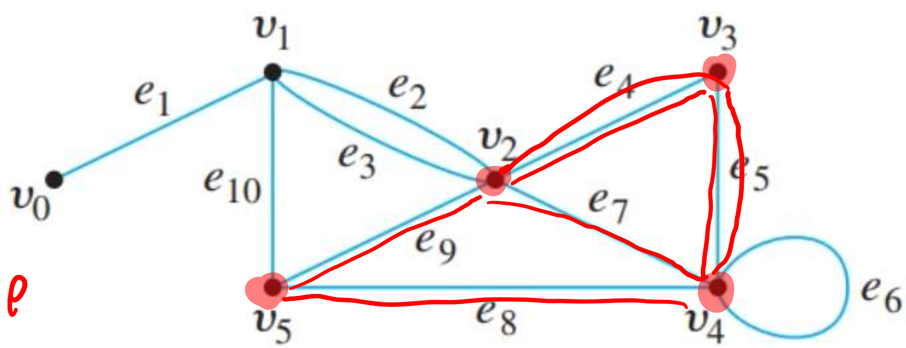
Trail ✓

not a path because of repeated vertex

Closed walk ✓

circuit ✓

not a simple circuit because v_4 appears twice

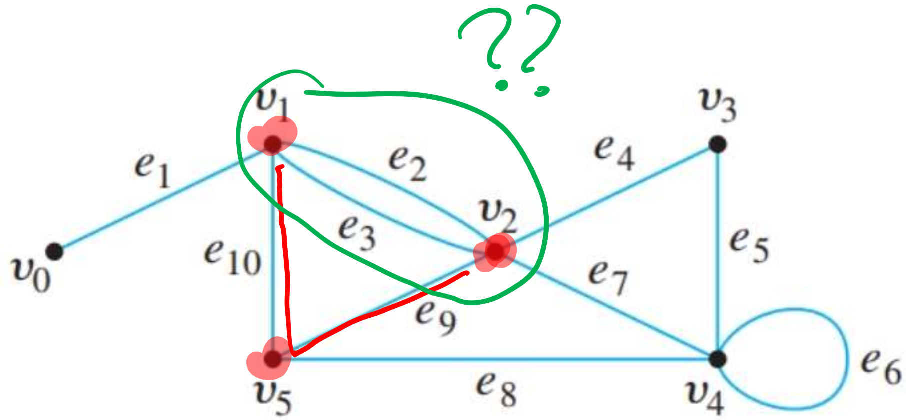


start ↓ stop ↓

(e) $v_2 v_3 v_4 v_5 v_2 v_4 v_3 v_2$

Closed walk (starts & stops at same vertex)

(Not a trail or a circuit because of repeated edges.)



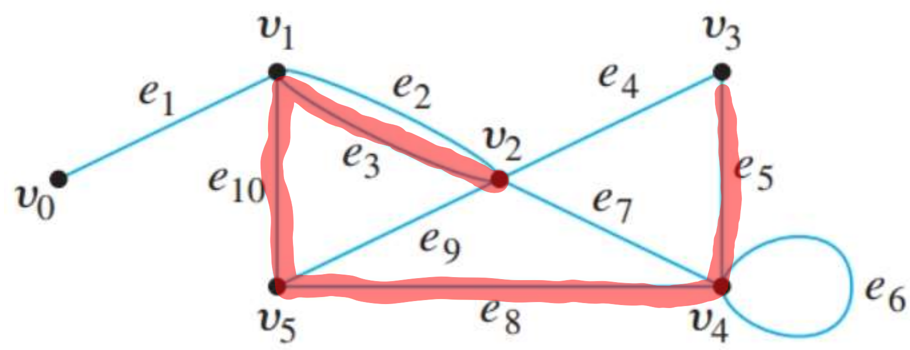
(f) $v_2 v_5 v_1 v_2$

ambiguous description

Could be $v_2 e_9 v_5 e_{10} v_1 e_2 v_2$

or $v_2 e_9 v_5 e_{10} v_1 e_3 v_2$

Not a valid description of a walk



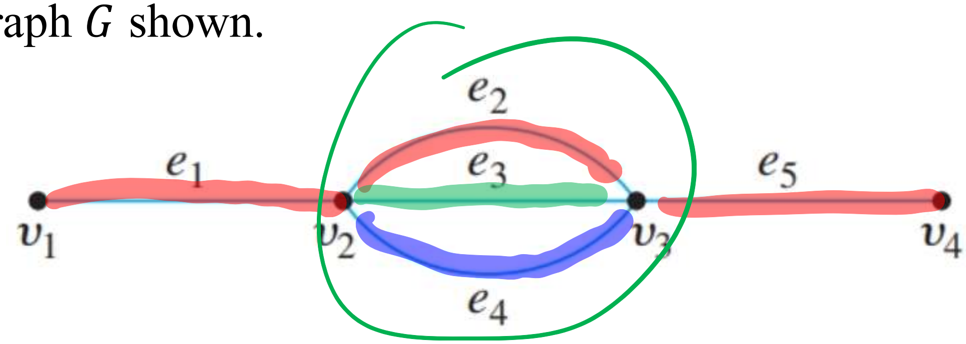
(g) $e_5 e_8 e_{10} e_3$

Shorthand for $v_3 e_5 v_4 e_8 v_5 e_{10} v_1 e_2$

Path no repeated edges or vertices

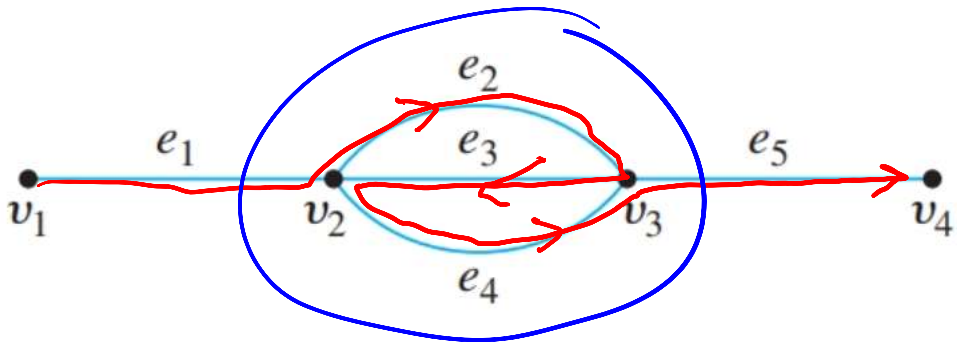
(not a closed walk or circuit because does not start &stop at same point.)

[Example 3] In the graph G shown.



(a) How many paths are there from v_1 to v_4?

3 paths



(b) How many trails are there from v_1 to v_4?

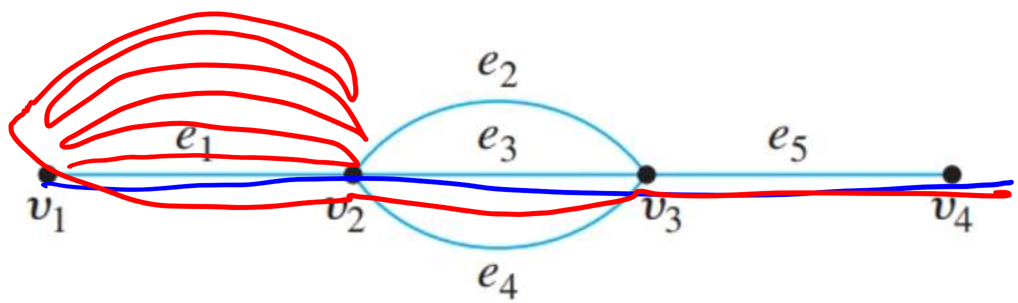
3 that go straight through

Trails that backtrack

Task	description	number
#1	Chose edge for $v_2 \to v_3$ first time.	$n_1 = 3$
#2	Choose edge for $v_3 \to v_2$	$n_2 = 2$
#3	Choose edge for $v_2 \to v_3$ second time.	$n_3 = 1$

Total number of ways $n = n_1 \cdot n_2 \cdot n_3 = 3 \cdot 2 \cdot 1 = 6$

Using addition rule Total number of trails is $3 + 6 = 9$



(c) How many walks are there from v_1 to v_4?

Infinite collection of possible walks

Subgraphs, Connected Vertices, Connected Graphs

Definition of **Subgraph**

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Definition of **Connected**

Let G be a graph.

Words: Two vertices v and w of G are connected.

Meaning: There is a walk from v to w.

Words: The graph G is connected.

Meaning: Given any two vertices v and w in G, there is a walk from v to w.

Meaning in Symbols: $\forall v, w \in V(G)(\exists \text{ a walk from } v \text{ to } w)$

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Lemma 10.1.1 about Connected Graphs

Let G be a graph.

a. If G is connected, then any two distinct vertices of G can be connected by a path.

b. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

c. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

Definition of **Connected Component**

Words: Graph H is a **connected component** of graph G.

Meaning: H has the following three properties

1. H is a subgraph of G.
2. H is connected.
3. No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

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[Example 4] Find the connected components for the graph shown below.