Video for Homework H10.1b Euler Circuits, Euler Trails, Hamiltonian Circuits

Reading: From Chapter 10 Theory of Graphs and Trees

- Section 10.1 Trails, Paths, Circuits
    - pages 684 – 692, Examples 10.1.6 – 10.1.10

Homework: H10.1b: 10.1 # 9,15,17,20,28,28,35,42

Topics:

- **Euler Circuits**
- **Euler Trails**
- **Hamiltonian Circuits**

From Video for Homework H10.1: This table of terminolgy from Section 10.1

Definition of **Walk, Trail, Path, Closed Walk, Circuit, Simple Circuit**

Let G be a graph, and let v and w be vertices in G.

- A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form $v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$, where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1,2, \dots, n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.
- A **trail from v to w** is a walk from v to w that does not contain a repeated edge.
- A **path from v to w** is a trail that does not contain a repeated vertex.
- A **closed walk** is a walk that starts and ends at the same vertex.
- A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.
- A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Also from Video for Homework H10.1:

This Summary of Definitions of Walk, Trail, Path, Closed Walk, Circuit, Simple Circuit

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Vertex?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed Walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	yes	yes

Euler Circuits

Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

It is helpful to view in one table the kinds of walks that we have learned about so far that must start and end at the same vertex.

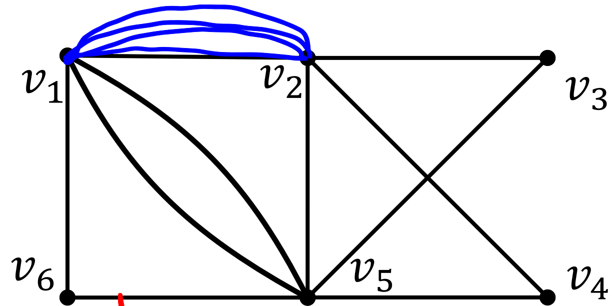
	Must Use Every Edge?	Repeated Edge?	Must use Every Vertex?	Repeated Vertex?	Starts, Ends at the Same Vertex?	Must Contain at Least One Edge?
Closed Walk	no	allowed	no	allowed	yes	no
Circuit	no	no	no	allowed	yes	yes
Simple Circuit	no	no	no	first and last only	yes	yes
Euler Circuit	yes	no	yes	allowed	yes	yes

The important thing to remember is that

- A *closed walk* can have *no edges* and can have *repeated edges*.
- All the kinds of *circuits must contain at least one edge* and *can not have repeated edges*.

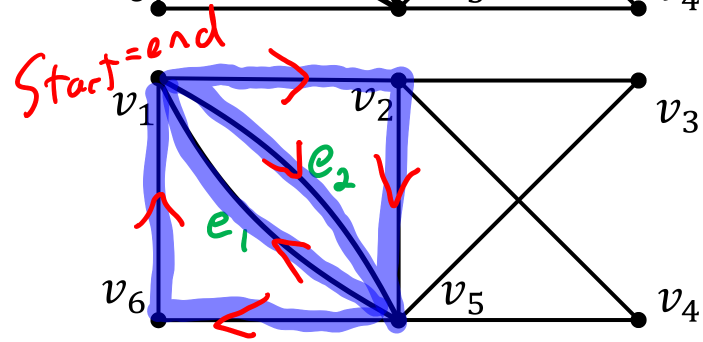
[Example 1] For the given graph, draw a walk of the indicated type.

Closed Walk



$v_1 v_2 v_1 v_2 v_1$

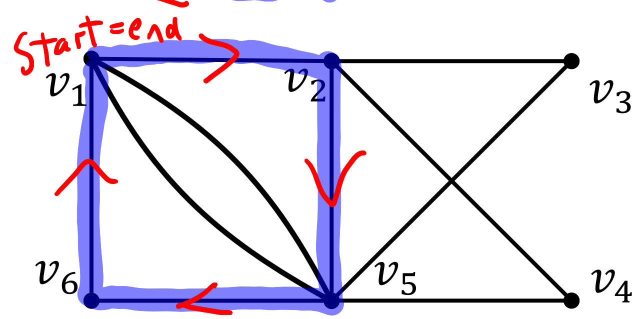
Circuit



$v_1 v_2 v_5 v_1 v_5 v_6 v_1$

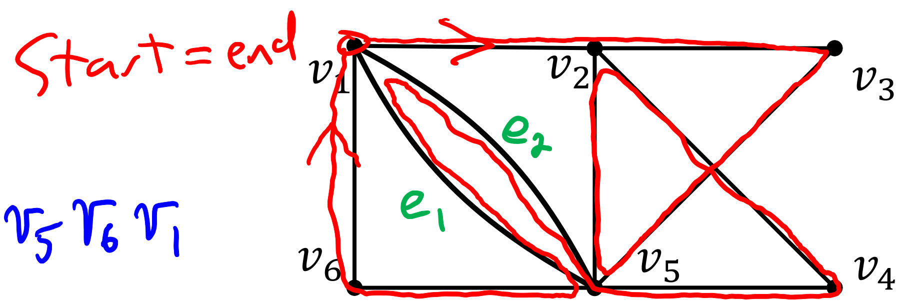
$v_1 v_2 v_5 e_1 v_1 e_2 v_5 v_6 v_1$

Simple Circuit



$v_1 v_2 v_5 v_6 v_1$

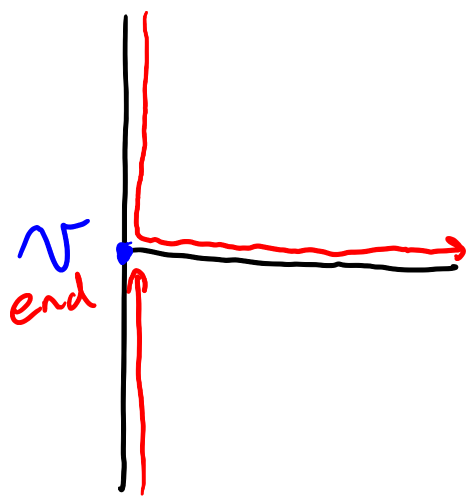
Euler Circuit



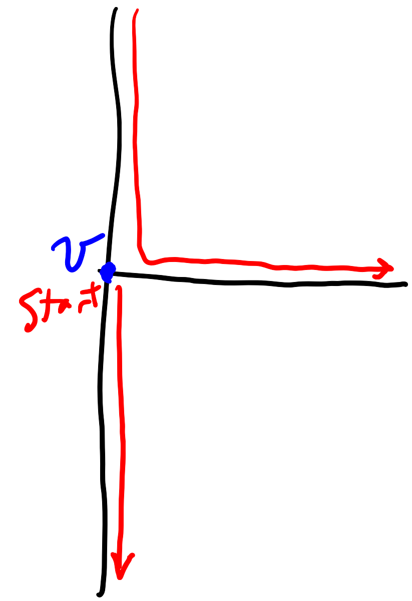
$v_1 v_2 v_3 v_5 v_2 v_4 v_5 e_2 v_1 e_1 v_5 v_6 v_1$

(and has a requirement that every edge must be used)

Observation: If a variety of walk has a requirement that repeated edges are *not* allowed, then any vertex that has an odd degree must be either the start point or the end point of the walk (**Exclusive OR**: the vertex cannot be both the start point and the end point.)



or maybe



Theorem 10.1.2

If a graph has an Euler circuit,

then every vertex of the graph has positive even degree.

Contrapositive Version

If some vertex of a graph has odd degree

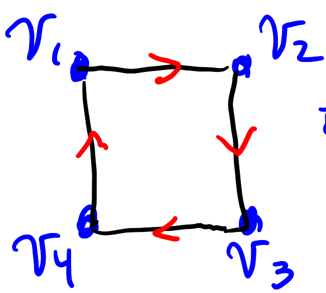
then the graph does not have an Euler circuit.

[Example 2] Do these graphs have an Euler Circuit?

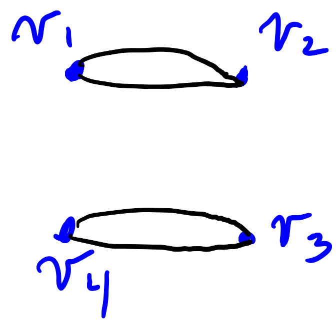
(a) G is a connected graph with seven vertices of degrees 1,1,2,2,3,3,6.

G does not have an Euler circuit because it has a vertex of odd degree.

(b) G is a graph with four vertices of degrees each of degree 2. Not enough information

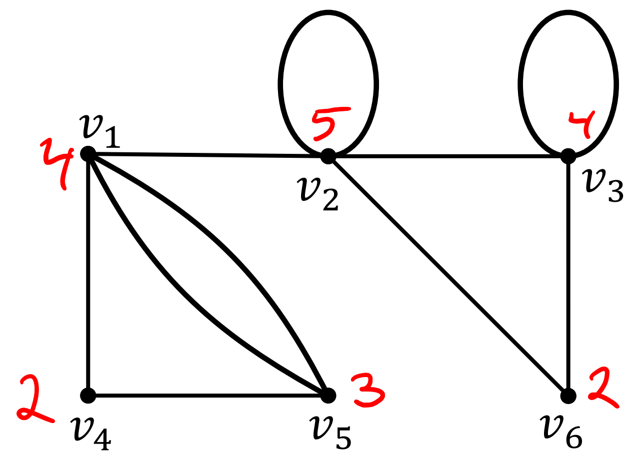


Euler circuit
$v_1 v_2 v_3 v_4$



no Euler circuit

(c) G is the graph shown at right.



Since there is a vertex that has odd degree, there is no Euler Circuit

What if a graph does not flunk the two obvious requirements? That is, what if the graph is connected and has all vertices of positive, even degree. Is that enough to guarantee that the graph will have an Euler circuit? That is the subject of Theorem 10.1.3

Theorem 10.1.3

If a graph G is connected and every vertex of G has positive, even degree, then G has an Euler circuit.

[Example 3](similar to 10.1#9)

A connected graph G has ten vertices with degrees 2,2,4,4,4,4,4,4,4

Does the graph have an Euler Circuit?

Yes, Note that G is connected ✓
every vertex of G has positive even degree ✓
So by Theorem 10.1.3, G has an Euler Circuit.

The Proof of Theorem 10.1.3 is in the form of an ***algorithm for constructing an Euler Circuit for a graph G in which the degree of every vertex is a positive, even integer***.

connected

The algorithm is shown on the next page.

Algorithm for constructing an Euler Circuit (graph G described on previous page)

Step 1: Pick any vertex v of G at which to start.

Step 2: Pick any sequence of adjacent vertices and edges, starting and ending at v and never repeating an edge. Call the resulting circuit C

Step 3: Check whether C contains every edge and vertex of G. If so, C is an Euler circuit, and we are finished. If not, perform the following steps.

Step 3a: Remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Call the resulting sub-graph G'.

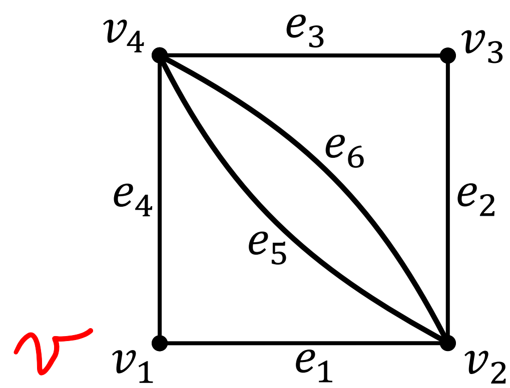
Step 3b: Pick any vertex w common to both C and G'.

Step 3c: Pick any sequence of adjacent vertices and edges of G', starting and ending at w and never repeating an edge. Call the resulting circuit C'.

Step 3d: Patch C and C' together to create a new circuit C'' as follows: Start at v and follow C all the way to w. Then follow C' all the way back to w. After that, continue along the untraveled portion of C to return to v.

Since the graph G is finite, execution of the steps outlined in this algorithm must eventually terminate. At that point an Euler circuit for G will have been constructed.

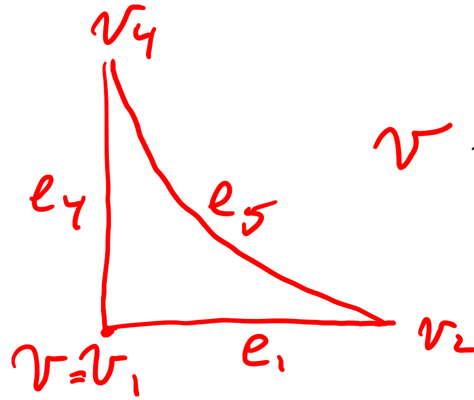
[Example 4] Show how the Algorithm works to construct an Euler Circuit for graph G shown.



Step 1 let vertex v be v_1

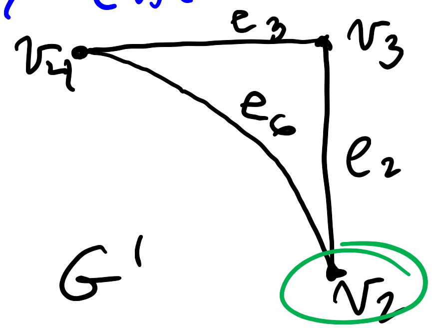
Step 2 Let Circuit C be

$v_1 e_1 v_2 e_5 v_4 e_4 v_1$



Step 3 Observe that C does not contain all of the edges and vertices of G. C is not an Euler circuit

Step 3a Graph G' will be as shown



Step 3b Let w = vertex v_2

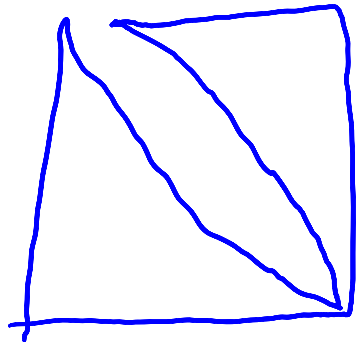
Step 3c Let Circuit C' be

$v_2 e_2 v_3 e_3 v_4 e_6 v_2$

Step 3d Create Circuit C''

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_6 v_2 e_5 v_4 e_4 v_1$

$v \quad w \quad w \quad v$



Go back to Step 3 again

Observe that circuit C'' contains every edge and every vertex of G. So C'' is an <u>Euler circuit</u>.

We're done!

Euler Trails

Definition

Let G be a graph and let v, w be two *distinct* vertices of G. An **Euler trail from $\boldsymbol{v}$ to $\boldsymbol{w}$** is a sequence of adjacent edges and vertices that starts at v, passes through every vertex of G *at least once*, uses every edge of G *exactly once*, and ends at w.

It is helpful to view in one table the kinds of walks that we have learned about so far that can start and end at different vertices.

	Must Use Every Edge?	Repeated Edge?	Must use Every Vertex?	Repeated Vertex?	Starts, Ends at the Same Vertex?	Must Contain at Least One Edge?
Walk	no	allowed	no	allowed	allowed	no
Trail	no	no	no	allowed	allowed	no
Euler Trail	yes	no	yes	allowed	no	yes
Path	no	no	no	no	no	no

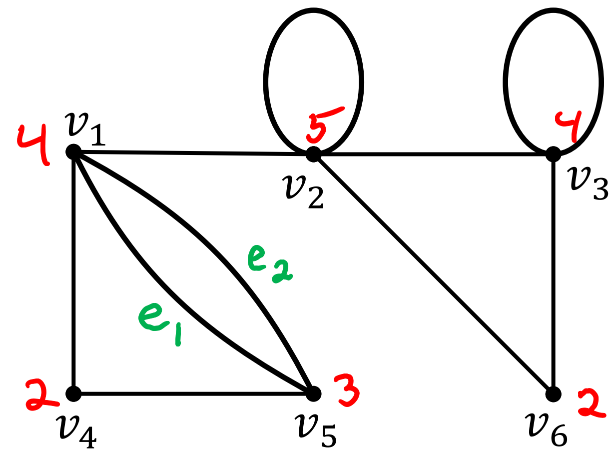
There is a useful criterion for knowing when a graph does or does not have an Euler Trail.

Corollary 10.1.5 Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

[Example 5]

(a) Does the graph shown have an Euler trail?

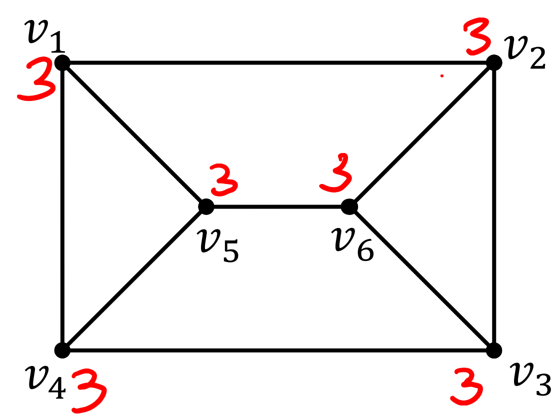
If so, describe one. If not, explain why not.



$v_5\, e_1\, v_1\, e_2\, v_5\, v_4\, v_1\, v_2\, v_2\, v_3\, v_3\, v_6\, v_2$

(b) Does the graph shown have an Euler trail?

If so, describe one. If not, explain why not.



Does not have an Euler Trail because too many vertices of odd degree!

Hamiltonian Circuits

Definition

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Remark:Note that although an Euler circuit for a graph G must include every vertex of G, it may visit some vertices more than once and hence may not be a Hamiltonian circuit. On the other hand, a Hamiltonian circuit for G does not need to include all the edges of G and hence may not be an Euler circuit.

It is helpful to view in one table the kinds of walks that we have learned about so far that must start and end at the same vertex.

	Must Use Every Edge?	Repeated Edge?	Must use Every Vertex?	Repeated Vertex?	Starts, Ends at the Same Vertex?	Must Contain at Least One Edge?
Closed Walk	no	allowed	no	allowed	yes	no
Circuit	no	no	no	allowed	yes	yes
Simple Circuit	no	no	no	first and last only	yes	yes
Euler Circuit	yes	no	yes	allowed	yes	yes
Hamiltonian Circuit	no	no	yes	first and last only	yes	yes

An important thing to remember is that a *closed walk allows repeated edges*. All the various kinds of *circuits do not allow repeated edges*.

There is a useful list of properties of graphs that have Hamiltonian Circuits.

Proposition 10.1.6 If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

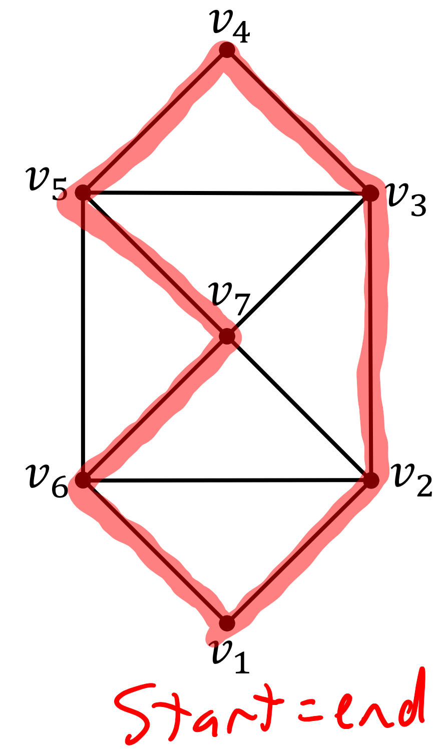
1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2

[Example 6]

(a) Does the graph shown have a Hamiltonian Circuit?

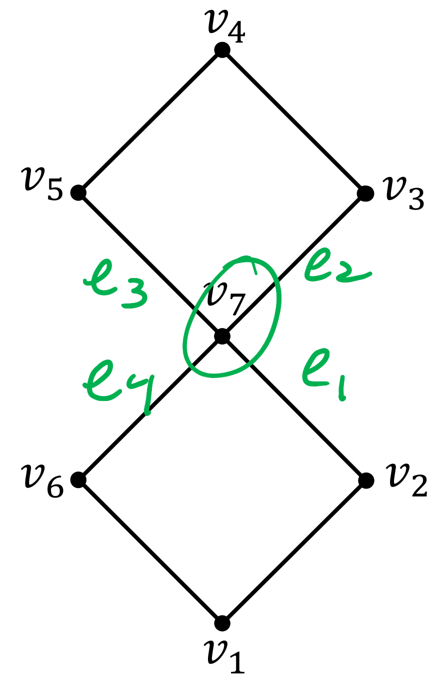
If so, describe one. If not, explain why not.

$v_1 v_2 v_3 v_4 v_5 v_7 v_6 v_1$



(b) Does the graph shown have a Hamiltonian Circuit?

If so, describe one. If not, explain why not.



No. If vertex v_7 is part of a subgraph H in which $\text{degree}(v_7) = 2$, then two of the edges $e_1\ e_2\ e_3\ e_4$ must not be included in H. This will result in either H not being connected or in two vertices of H not having degree 2.