Video for Homework H10.2a Matrix Representations of Graphss

Reading: From Chapter 10 Theory of Graphs and Trees

- Section 10.2 Matrix Representations of Graphs
  - pages 698 - 703, Examples 10.2.1 – 10.2.5

Homework: H10.2a: 10.2 # 2,5,6

Topics:

- **Review of Matrix Terminology**
- **The Adjacency Matrix Corresponding to a Directed Graph**
- **The Adjacency Matrix Corresponding to an Undirected Graph**
- **Matrix Block Forms Corresponding to Connected Components of Graphs**

Section 10.2 begins with a review of Matrix notation and terminology

Definition of Matrix

Printed Words: An $m \times n$ matrix A over a set S

Spoken Words: An m by n matrix A over a set S

n columns

Symbol: $A = (a_{ij})$

Meaning: a rectangular array of elements of S arranged into m rows and n columns

m rows

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Additional Matrix Terminology and Notation

The ***i*^*th* row of *A*** is $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{bmatrix}$ 1 x n matrix

The ***j*^*th* column of *A*** is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$ m x 1 matrix

The entry a_{ij} in the i^{th} row and j^{th} column of A is called the ***ij*^*th* entry of *A***.

An $m \times n$ matrix is said to have **size $\boldsymbol{m} \times \boldsymbol{n}$**. (spoken m by n).

To say that two matrices are **equal** means that they have the same size and all their corresponding entries are equal. That is, $a_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$

A matrix for which the numbers of rows and columns are equal is called a **square matrix**.

If A is a square matrix of size $n \times n$, then the **main diagonal of *A*** consists of all the entries

$$a_{11}, a_{22}, \dots, a_{nn}$$

To say that an $n \times n$ square matrix $A = \left(a_{ij}\right)$ is **symmetric** means that

$$a_{ij} = a_{ji} \text{ for all } i, j = 1, \dots, n$$

The Matrix Corresponding to a Directed Graph

Definition of Adjacency Matrix

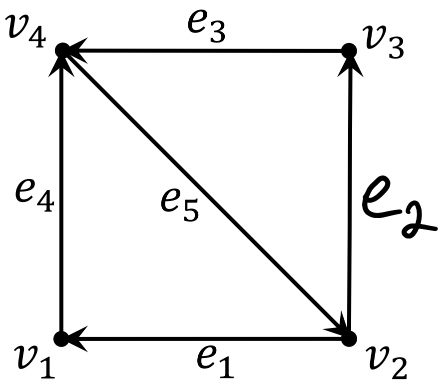
Words: The **adjacency matrix of *G***

Usage: G is a directed graph with ordered vertices$v_1, v_1, \ldots v_n$

Meaning: The $n \times n$ matrix $A = (a_{ij})$ over the set of nonnegative integers defined by

$$a_{ij} = \text{ the number of arrows from } v_i \text{ to } v_j \qquad \text{for all } i, j = 1,2, \ldots, n$$

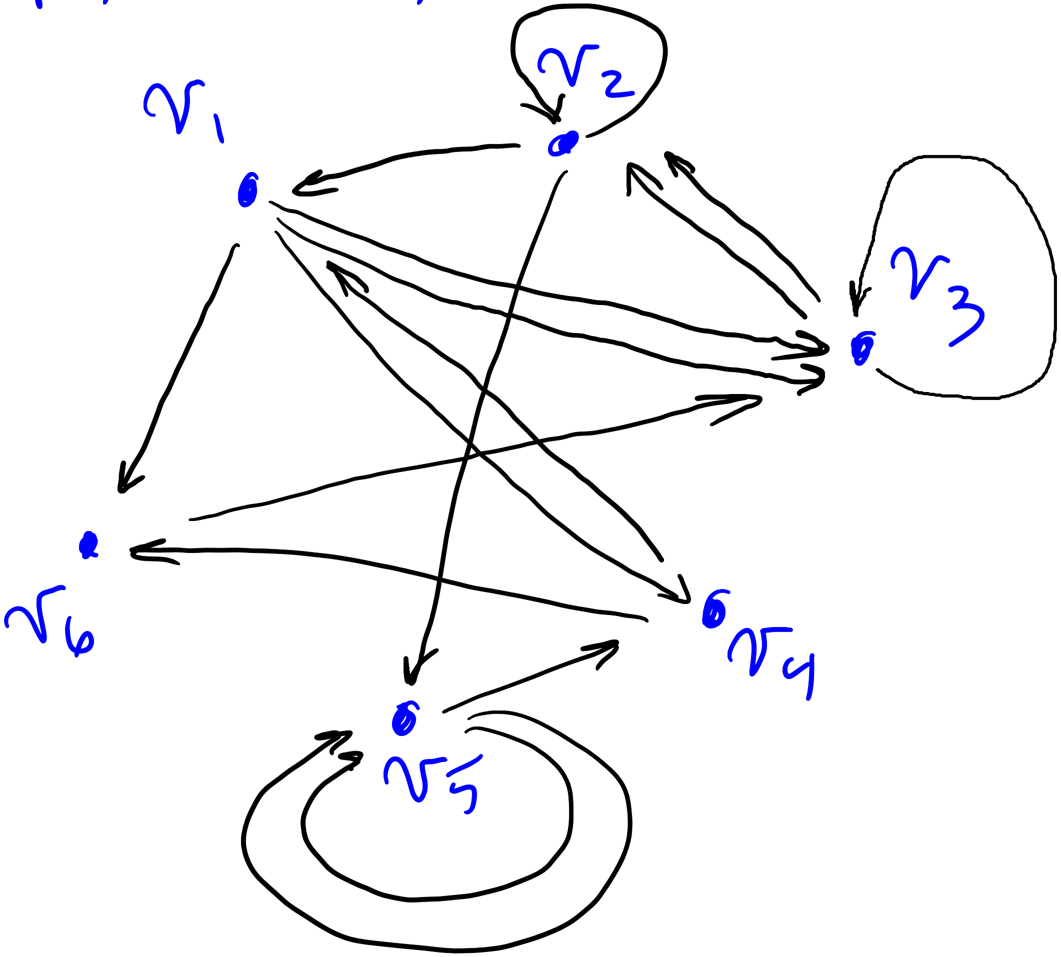
[Example 1] Find the adjacency matrix for the directed graph.



$$\begin{array}{cc|cccc|} & & & j & & \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 0 & & 1 \\ i & 2 & 1 & 0 & 1 & 0 \\ & 3 & 0 & 0 & 0 & 1 \\ & 4 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

[Example 2] Draw the directed graph for the adjacency matrix $A = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

Matrix is 6x6, so we need 6 vertices



(Fixed mistake in video)

The Matrix Corresponding to an Undirected Graph

Definition of Adjacency Matrix

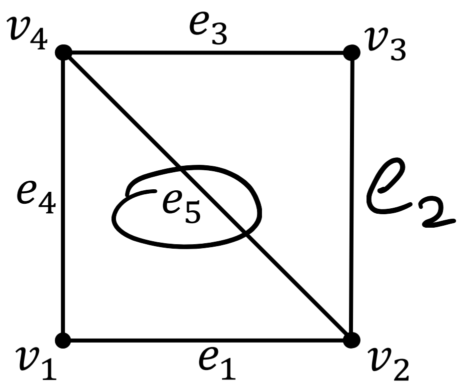
Words: The **adjacency matrix of *G***

Usage: G is an undirected graph with ordered vertices$v_1, v_1, \dots v_n$

Meaning: The $n \times n$ matrix $A = (a_{ij})$ over the set of nonnegative integers defined by

$$a_{ij} = \text{ the number of edges connecting } v_i \text{ and } v_j \qquad \text{for all } i, j = 1,2, \dots, n$$

[Example 3] Find the adjacency matrix for the undirected graph.



4 vertices, so we need a 4x4 matrix

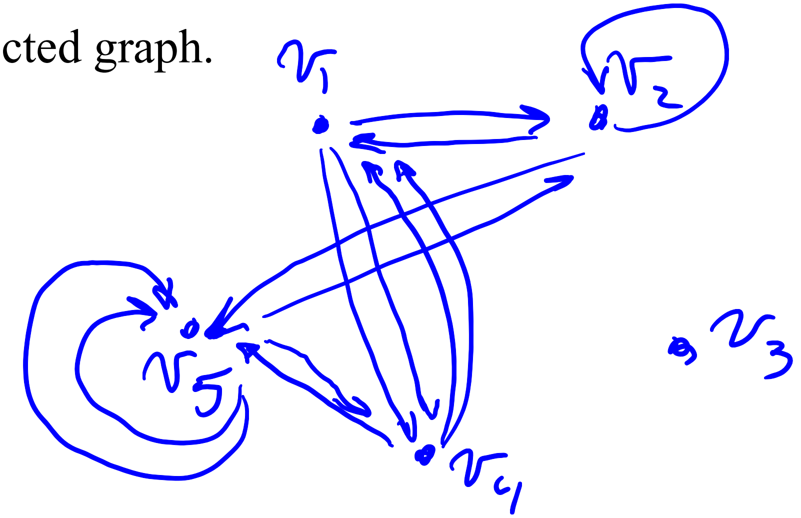
$$\begin{array}{ccc}
 & & j \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
i & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}$$

Observation:

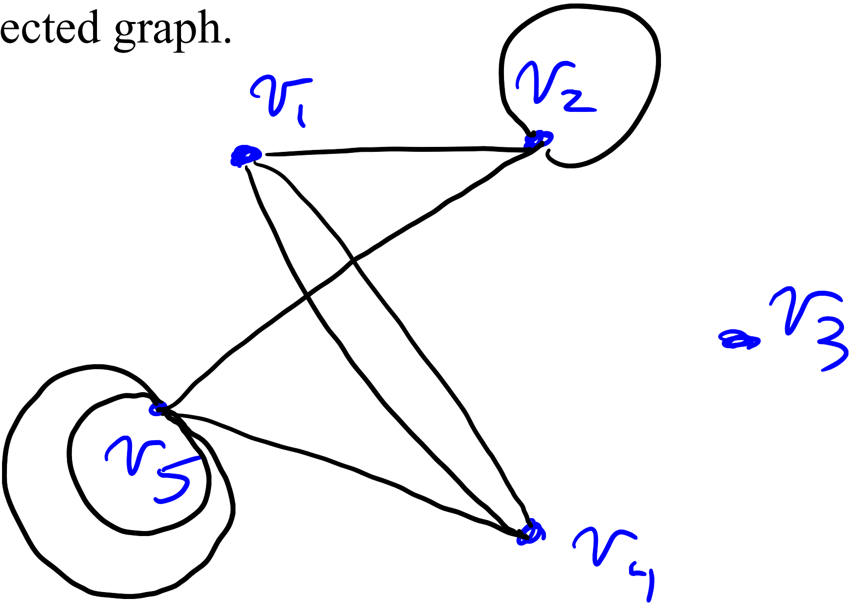
- Undirected graphs will always have adjacency matrices that are symmetric
- When drawing a graph that corresponds to a given adjacency matrix,
  - If the matrix is not symmetric, then you know that the graph must be a directed graph.
  - If the matrix is symmetric, then there is a corresponding directed graph and also a corresponding undirected graph.

[Example 4] For the given symmetric adjacency matrix $A = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$

(a) Draw the corresponding directed graph.

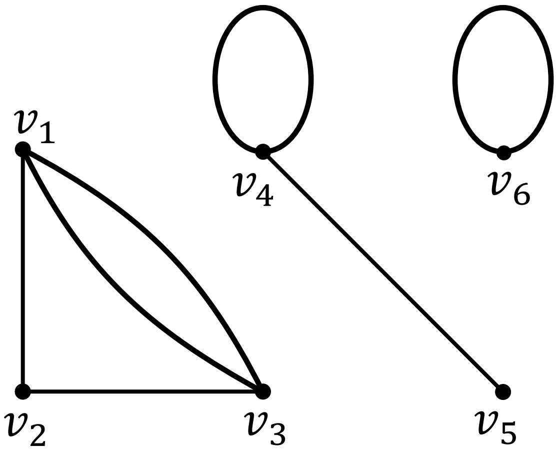


(b) Draw the corresponding undirected graph.



The Matrix Block Forms Corresponding to Connected Components of Graphs

[Example 5] Find the adjacency for the given graph.



Six vertices, so need 6x6 matrix

$$\begin{array}{cc|cccccc}
 & & & & & j & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \\
 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\
i & 4 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & 5 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 6 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

Theorem 10.2.1 Matrix Block Forms Corresponding to Graph Connected Components

If G is a graph with connected components $G_1, G_2, G_3, \dots, G_k$ and if each connected component G_i contains n_i vertices and if all of the vertices of G are numbered consecutively in the manner shown

$$\underbrace{v_1, v_2, \dots, v_{i_1}}_{\in G_1}, \underbrace{v_{i_1+1}, \dots, v_{i_1+i_2}}_{\in G_2}, \underbrace{v_{i_1+i_2+1}, \dots, v_{i_1+i_2+i_3}}_{\in G_3}, \dots, \underbrace{v_{i_1+i_2+\cdots+i_{k-1}+1}, \dots, v_{i_1+i_2+\cdots+i_{k-1}+i_k}}_{\in G_k}$$

then the adjacency matrix of G has the form

$$A = \begin{bmatrix} A_1 & O & O & \cdots & O \\ O & A_2 & O & \cdots & O \\ O & O & A_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_k \end{bmatrix}$$

where each A_i is the $n_i \times n_i$ (square) adjacency matrix of G_i, for $i = 1,2,3, \dots, k$ and the O symbols represent matrices whose entries are all 0

Remark 1 on Theorem 10.2.1

Each of the matrices A_i, for $i = 1,2,3, \dots, k$, will be a squre matrix, but the matrices represented by the symbols O are usually not square and usually not the same shape. The only time the matrices represented by the O symbols are *square* is when all of the the matrices A_i, for $i = 1,2,3, \dots, k$ are the *same size*. That is, when

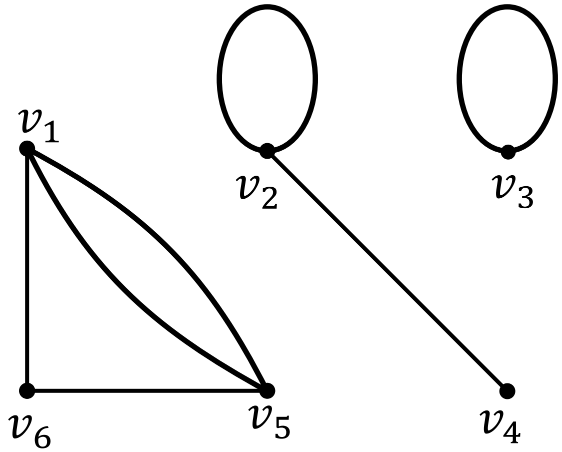
$$n_1 = n_2 = n_3 = \cdots = n_k = n.$$

In this special case, not only will all of the matrices represented by the O symbols be *square*, they will also all be the same size, $n \times n$.

Remark 2 on Theorem 10.2.1

It is crucial that the vertices be numbered in a way that sequential groups of vertex numbers correspond to connected components of the graph. If this numbering scheme is not followed, then the corresponding adjacency matrix will not have a block form as described in the theorem.

[Example 6] Find the adjacency for the graph from previous **[Example 5]** with this different vertex numbering.



$$
i \quad \begin{array}{c|cccccc}
 & & & j & & & \\
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 0 & 0 & 0 & 2 & 1 \\
2 & 0 & 1 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 2 & 0 & 0 & 0 & 0 & 1 \\
6 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

matrix does not have the block form described in the theorem

Remark 3 on Theorem 10.2.1

If an adjacency matrix does not have the block form described in the theorem, that does not necessarily indicate that the corresponding graph is a connected graph.

Consider, for example, the matrix given in **[Example 4].** That matrix does not have the block form described in the theorem, but the corresponding undirected graph has two connected components. Of course, if the vertices of corresponding graph were renumbered as described in the theorem, then the adjacency matrix *would* have the block form described in the theorem.

Remark 4 on Theorem 10.2.1

- If any of the matrices A_i, for $i = 1,2,3, \dots, k$, is *asymmetric*, then the matrix A can only correspond to a *directed graph*.
- If all of the matrices A_i, for $i = 1,2,3, \dots, k$, are *symmetric*, then there is both a *directed graph* corresponding to matrix A and an *undirected graph* corresponding to matrix A.

[Example 6] Draw an undirected graph for the adjacency matrix $\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

6x6 matrix, so we need 6 vertices