Axioms, Definitions, and Theorems Through Chapter 6

2021 – 2022 Spring Semester MATH 3110/5110

Definition of Abstract Geometry (Barsamian's version, correcting an error in the book's definition)

An *abstract geometry* $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called *points* and $\mathcal{L}$ denotes a *non-empty* set whose elements are called *lines*, which are *sets of points* satisfying the following two requirements, called *axioms*:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in \mathcal{L}$ there exist **at least two** distinct points that are elements of the line.

Definition of Incidence Geometry

An *incidence geometry* is an ordered pair $(\mathcal{P}, \mathcal{L})$ that is satisfies all the requirements of an *abstract geometry* and also satisfies the following two additional *axioms*:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist **(at least) three** non-collinear points.

Definition of Notation for the Unique Line Containing Two Given Points

Symbol: $\overleftrightarrow{AB}$

Spoken: *line A B*

Usage: There is an *incidence geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ in the discussion and $A, B \in \mathcal{P}$ are two distinct points

Meaning: the unique line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,

If $l_1 \cap l_2$ has two or more distinct points,

then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

Corollary 2.1.7 (contrapositive of Theorem 2.1.6)

Given two lines l_1 and l_2 in an *incidence geometry*,

If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),

then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P,Q) \geq 0)$

(ii) $d(P,Q) = 0$ if and only if $P = Q$

(iii) $d(P,Q) = d(Q,P)$

Definition of a Ruler for a Line

words: f is a ruler for line l. (**alternate words:** f is a coordinate function for line l.)

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a *bijection*.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P,Q)$$

Additional Terminology:

The equation above is called the ***Ruler Equation***.

The number $f(P)$ is called the ***coordinate of P with respect to f***.

Definition of Metric Geometry

A *metric geometry* $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$.
- Every line $L \in \mathcal{L}$ has a *ruler*. (This requirement is called the ***Ruler Postulate***.)

Theorem 2.3.2 (Ruler Placement Theorem)

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- distinct points $A, B \in \mathcal{P}$

Claim: There exists a ruler g for line $\overleftrightarrow{AB}$ such that $g(A) = 0$ and $g(B) > 0$.

Terminology: Such a ruler g is called a ***ruler with A as origin and B positive***.

Definition of Betweenness for Real Numbers

Symbol: $x * y * z$

Spoken: y *is between* x *and* z.

Usage: $x, y, z \in \mathbb{R}$

Meaning: $x < y < z$ or $z < y < x$

Remark: It is a property of real numbers that for given any three distinct real numbers, one is smallest, one is largest, and the other is between them

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: B *is between* A *and* C.

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

Theorem 3.2.2 (Really a Corollary of the Definition)

Given: Points A, B, C in a metric geometry

Claim: The following are equivalent (*TFAE*): (i) $A - B - C$

(ii) $C - B - A$

Theorem 3.2.3 Betweenness of Points is Related to Betweenness of Coordinates

Given: Collinear points A, B, C on line l with ruler f in a metric geometry

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$ (betweenness of *points*)

(ii) $f(A) * f(B) * f(C)$ (betweenness of *coordinates*)

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

Claim: Exactly one of the points is between the other two.

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim: (i) There exists a point C with $A - B - C$

(ii) There exists a point D with $A - D - B$

Definition of Segment

Symbol: $\overline{AB}$

Spoken: *segment* $A\ B$.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

Symbol: length$(\overline{AB})$

Spoken: the **length** of segment $\overline{AB}$

Meaning: the number AB. That is, the length is the number $d(A, B)$.

Definition of Ray

Symbol: $\overrightarrow{AB}$

Spoken: *ray* $A\ B$.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overrightarrow{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\}$$
$$= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}$$

Additional Terminology

The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

The **interior of the ray** is the set of all points of the ray except the initial point:

$$\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$

Theorem 3.3.4 Subtlety in the Notation for a Ray

(i) (Different symbols that represent the same ray.) If $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.

(ii) (If two rays are equal then their initial points are equal.) If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

Theorem: Existence and Uniqueness of the Midpoint of a Segment

If A, B are distinct points in a metric geometry, then segment $\overline{AB}$ has exactly one midpoint.

Definition of Angle

Symbol: $\angle ABC$

Spoken: *angle* $A\ B\ C$.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$

Additional Terminology: The **vertex** of $\angle ABC$ is the point B.

Definition of Triangle

Symbol: ΔABC

Spoken: *triangle* $A\ B\ C$.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

Additional Terminology:

The **vertices** of ΔABC are the points A, BC.

The **sides** (or **edges**) of ΔABC are the segments $\overline{AB}, \overline{BC}, \overline{CA}$.

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \ldots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. That is,

$$\text{If } i \neq j \text{ then } A_i \cap A_j = \phi$$

Definition of Convex

Words: S is convex

Usage: A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.

Meaning: for every two distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.

Quantified version: $\forall A, B \in S, A \neq B (\overline{AB} \subset S)$.

Universal Conditional Version: $\forall A, B \in \mathcal{P}, A \neq B (\text{If } A, B \in S \text{ then } \overline{AB} \subset S)$

Definition: The Plane Separation Axiom (PSA) (**Barsamian's version of the definition**)

Words: A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)

Meaning: For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

(i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

(ii) Each of the *half planes* is convex.

(iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.

Additional Terminology:

Line l is called the ***edge*** of *half planes* H_1 and H_2.

Words: *Points A, B lie on the **same side** of line l.*

Meaning: Points A, B are elements of the same half plane associated to l.

Words: *Points A, B lie on **opposite sides** of line l.*

Meaning: Points A, B are elements of different half planes associated to l.

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapos) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.

Definition: Pasch's Postulate (PP)

Words: A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies **Pasch's Postulate (PP)**

Meaning: For every line and for every triangle, if the line intersects a side of the triangle at a point that is not a vertex, then the line intersects at least one of the opposite sides.

Theorem About Two Equivalent Statements in a Metric Geometry (Contains all the information of the textbook's Theorems 4.3.1 and 4.3.2, and 4.3.3, and can be used in place of any of those three theorems.)

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

Claim: The following statements are equivalent (TFAE)

(1) The metric geometry satisfies the *Plane Separation Axiom (PSA)*.

(2) The metric geometry satisfies *Pasch's Postulate (PP)*.

Definition of Pasch Geometry

A ***Pasch Geometry*** is a *metric geometry* that satisfies the *Plane Separation Axiom (PSA)*.

Remark: By the *Theorem About Two Equivalent Statements in a Metric Geometry*, we see that Pasch Geometries are also the metric geometries that satisfy *Pasch's Postulate (PP)*.

Definition: Peano's Axiom (PA)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies ***Peano's Axiom (PA)***
- **Meaning:** Given triangle ΔABC and points D, E such that $B - C - D$ and $A - E - C$, there exists a point $F \in \overleftrightarrow{DE}$ such that $A - F - B$ and $D - E - F$.

Theorem (The statement proven in Book Exercise 4.3#1): *PSA* ➔ *PA*

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

Claim: If the metric geometry satisfies the *Plane Separation Axiom (PSA)*,

then the metric geometry satisfies *Peano's Axiom (PA)*.

Theorem 4.4.1 In a Pasch Geometry,

if $\mathcal{A}$ is a nonempty convex set that does not intersect line l,

then all points of $\mathcal{A}$ lie on the same side of l.

Theorem 4.4.2 In a Pasch Geometry,

let $\mathcal{A}$ be a line, ray, segment, interior of a ray, or interior of a segment.

(i) If l is a line with $\mathcal{A} \cap l = \phi$, then all of $\mathcal{A}$ lies on one side of l.

(ii) If $A - B - C$ and $\overleftrightarrow{AC} \cap l = \{B\}$

then $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overline{BA})$ both lie on the same side of l,

while $\text{int}(\overrightarrow{BC})$ and $\text{int}(\overline{BC})$ both lie on the other side of l.

Theorem 4.4.3 (*The Z Theorem*) In a Pasch geometry, if P and Q are on opposite sides of $\overleftrightarrow{AB}$,

then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \phi$. In particular, $\overline{BP} \cap \overline{AQ} = \phi$.

Definition of More Descriptive Half Plane Notation

Symbol: $H_{\overleftrightarrow{AB},C}$

Usage: A, B, C are non-collinear points in a Pasch geometry.

Meaning: The half plane of line A, B that contains C.

Definition of Angle and Triangle Interiors

Symbol: $\text{int}(\angle ABC)$

Spoken: *the interior of angle A, B, C*

Meaning: $H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},A}$

Symbol: $\text{int}(\Delta ABC)$

Spoken: *the interior of triangle* A, B, C

Meaning: $H_{\overleftrightarrow{AB},C} \cap H_{\overleftrightarrow{BC},A} \cap H_{\overleftrightarrow{CA},B}$

Theorem 4.4.6 Given $\angle ABC$ in a Pasch geometry, if $A - P - C$, then $P \in \text{int}(\angle ABC)$.

Immediate consequence (corollary) of Theorem 4.4.6: In a Pasch geometry, all points in the interior of one side of a triangle are in the interior of the opposite angle. That is, in any triangle ΔABC the following subset relationship is true: $\text{int}(\overline{AC}) \subset \text{int}(\angle ABC)$

Theorem 4.4.7 (The Crossbar Theorem)

In a Pasch geometry, if $P \in \text{int}(\angle ABC)$, then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F such that $A - F - C$.

Theorem (Converse of the Statement of the Crossbar Theorem) (Proven in Exercise 4.4#12

Given $\angle ABC$ and point P in a Pasch Geometry, if $\overrightarrow{BP}$ intersects $\text{int}(\overline{AC})$, then $P \in \text{int}(\angle ABC)$.

Definition of Quadrilateral

words: quadrilateral A, B, C, D

symbol: $\square ABCD$

usage: A, B, C, D are distinct points, no three of which are collinear, and such that the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ intersect only at their endpoints.

meaning: quadrilateral A, B, C, D is the set $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$

additional terminology: Points A, B, C, D are each called a ***vertex*** of the quadrilateral. Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CD}$ and $\overline{DA}$ are the ***sides*** of the quadrilateral. Segments $\overline{AC}$ and $\overline{BD}$ are the ***diagonals***.

In a *Pasch geometry*, a ***"convex-shape quadrilateral"*** is one in which all the points of any given side lie on the same side of the line determined by the opposite side. A quadrilateral that does not have this property is called a ***"non-convex-shape quadrilateral"***. (Note: the book says "convex quadrilateal" and "non-convex quadrilateral.)

Definition: The symbol $\mathcal{A}$ denotes the set of all angles in a Pasch Geometry

Definition of Angle Measure

Words: Angle measure (or protractor) based on r_0

Usage: There is a Pasch geometry in the discussion, and r_0 is a fixed positive real number

Meaning: a function $m: \mathcal{A} \to \mathbb{R}$ that has these three properties (the *Axioms of Angle Measure*)

(i) $0 < m(\angle ABC) < r_0$

(ii) (This statement is called the *Angle Constuction Axiom*)

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

(iii) (This statement is called the *Angle Addition Axiom*)

Angle measure is "additive" in the following sense:

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

The three *Axioms of Angle Measure* are a "wish list" of properties that a function must have in order to be qualified to be called an "angle measure".

Definition: A **Protractor Geometry** is an ordered quadruple $(\mathcal{P}, \mathcal{L}, d, m)$ such that the ordered triple $(\mathcal{P}, \mathcal{L}, d)$ is a *Pasch Geometry* and m is an *angle measure* for $(\mathcal{P}, \mathcal{L}, d)$.

Definition: Euclidean Angle Measure is the function $m_E: \mathcal{A} \to \mathbb{R}$ ($\mathcal{A}$ is the set of angles in $\mathbb{R}^2$) defined by

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A - B, C - B\rangle}{\|A - B\|\|C - B\|}\right)$$

Remarks: Using *Degrees* for the $\cos^{-1}$, because we are always using $r_0 = 180$ in our course.

The expression $A - B$ is the *Euclidean Vector from B to A*.

Definition of the Euclidean Tangent to a Poincaré Ray

For distinct points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ in $\mathbb{H}$, the **Euclidean Tangent to Poincaré Ray $\overrightarrow{PQ}$** is denoted by the symbol T_{PQ} and is computed as follows:

If $x_P = x_Q$, then $\overleftrightarrow{PQ}$ is a *type I line* ${}_{x_P}L$ and $T_{PQ} = (0, y_Q - y_P)$

If $x_P < x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = (y_P, c - x_P)$

If $x_P > x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = -(y_P, c - x_P)$

Observe that in the case that $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$, the computation of T_{PQ} only needs the value of c for the line. The value of r is not used for the computation of T_{PQ}, but it turns out that the value of r will always equal the norm, $\|T_{PQ}\|$. The norm is needed for the angle calculation. So it is worth finding both c and r for the line.

Definition: Poincaré Angle Measure is the function $m_E: \mathcal{A} \to \mathbb{R}$ ($\mathcal{A}$ denotes the set of angles in $\mathbb{H}$) defined by

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right)$$

Definition of the Side-Angle-Side Axiom

Words: *A protractor geometry satisfies the Side-Angle-Side (SAS) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Definition of Neutral Geometry

A **neutral geometry** (or **absolute geometry**) is a protractor geometry that satisfies *SAS.*

Proposition 6.1.2 (Euclidean Law of Cosines) (proven in exercise 5.4#3)

If P, Q, R are three non-collinear points in $\mathbb{R}^2$,

then $\left(d_E(P,R)\right)^2 = \left(d_E(Q,P)\right)^2 + \left(d_E(Q,R)\right)^2 - 2d_E(Q,P)d_E(Q,R)\cos\left(m_E(\angle PQR)\right)$.

In other words, for triangle ΔPQR, if p, q, r are defined to be the lengths of the sides opposite those vertices and $\theta = m_E(\angle PQR)$, then $q^2 = p^2 + q^2 - 2pq\cos(\theta)$

Propositions 6.1.3 *The Euclidean plane satisfies SAS (and therefore is a neutral geometry).*

Proposition 6.1.4 *The Poincaré plane satisfies SAS (and therefore is a neutral geometry).*

Pythagorean Theorem of Euclidean Geometry (proven in exercise 6.1#6, Corollary of the Law of Cosines)

In the Euclidean plane, if triangle ΔABC has a right angle at C, then $\left(d_E(A,B)\right)^2 = \left(d_E(C,B)\right)^2 + \left(d_E(C,A)\right)^2$

In other words, for triangle ΔABC, if a, b, c are the lengths of the sides opposite those vertices $\angle ACB$ is a right angle, then $c^2 = a^2 + b^2$.

Corollary (*SOHCAHTOA* interpretation of Sine, Cosine, Tangent) (proven in 6.1#7)

In the *Euclidean plane*, if triangle ΔABC has a right angle at C, and $m_E(\angle B) = \theta$, then

$$\sin(\theta) = \frac{AC}{AB} \text{ and } \cos(\theta) = \frac{BC}{BA} \text{ and } \tan(\theta) = \frac{AC}{BC}$$

Theorem 6.1.5 (*Pons Asinorum*) (*Isosceles Triangle Theorem*) (*CS* ➔ *CA* Theorem)

In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.

Definition of the Angle-Side-Angle Axiom

Words: *A protractor geometry satisfies the Angle-Side-Angle (ASA) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.2.1 A Neutral Geometry Satisfies the Angle-Side-Angle (*ASA*) Axiom

Theorem 6.2.2 (Converse of the Statement of Pons Asinorum) (CA ➔ CS Theorem)

In Neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if *CA* then *CS*.

Definition of the Side-Side-Side Axiom

Words: *A protractor geometry satisfies the Side-Side-Side (SSS) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.2.3: A Neutral Geometry Satisfies the Side-Side-Side (*SSS*) axiom.

Theorem 6.2.5 Existence of At Least One Perpendicular to a Line through a Point *Not On the Line*

In a neutral geometry, if B is a point not on a line l,

then there exists ***at least one*** line l' such that l' contains B and $l' \perp l$.

Theorem 6.3.1: In a metric geometry, the following are equivalent (TFAE)

(i) $AB < CD$

(ii) There exists a point $G \in \text{int}(\overline{CD})$ such that $\overline{AB} \simeq \overline{CG}$

Theorem 6.3.2: In a metric geometry, the following are equivalent (TFAE)

(i) $m(\angle ABC) < m(\angle DEF)$

(ii) There exists a point $G \in \text{int}(\angle DEF)$ such that $\angle ABC \simeq \angle DEG$

Theorem 6.3.3 (Neutral Exterior Angle Theorem) In a neutral geometry, the measure of an exterior angle of a triangle is greater than the measure of either of its remote interior angles.

Unstated Corollary of the Neutral Exterior Angle Theorem

In a neutral geometry, if B is a point not on a line l,

then there ***cannot be more than one*** line that contains B and is perpendicular to l.

Unstated Corollary of Theorem 6.2.5 and of the Neutral Exterior Angle Theorem

In a neutral geometry, if B is a point not on a line l,

then there exists ***exactly one*** line l' such that l' contains B and $l' \perp l$.

Corollary 6.3.4 In neutral geometry, given line l and point P, there is *exactly one* line l' such that $P \in l'$ and $l' \perp l$.

Definition of the Side-Angle-Angle Axiom

Words: *A protractor geometry satisfies the Side-Angle-Angle (SAA) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.3.5: A Neutral Geometry Satisfies the Side-Angle-Angle (*SAA*) axiom.

Theorem 6.3.6 (The *BS* ➔ *BA* Theorem): In Neutral Geometry, if one side of a triangle is longer than another side, then the angle opposite the longer side is bigger than the angle opposite the shorter side. That is, in a triangle, *if BS then BA*. In symbols, *BS* ➔ *BA*.

Theorem 6.3.7 (The *BA* ➔ *BS* Theorem): In Neutral Geometry, if one angle of a triangle is bigger than another angle, then the side opposite the bigger angle is longer than the side opposite the smaller angle. That is, in a triangle, *if BA then BS*. In symbols, *BA* ➔ *BS*.

Theorem 6.3.7 (The *BA* ➔ *BS* Theorem) (with vertices named)

(Original) In Neutral Geometry triangle ΔABC, if $m(\angle ACB) > m(\angle ABC)$ then $AB > AC$.

(Contrapositive) In Neutral Geometry triangle ΔABC, if $AB \ngtr AC$ then $m(\angle ACB) \ngtr m(\angle ABC)$.

Theorem 6.3.8 (The *Triangle Inequality of Neutral Geometry*)

Version without vertices named: In Neutral Geometry, the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.

Theorem 6.4.1 If a triangle has a right angle, then the other two angles are acute, and the longest side of the triangle will be the side opposite the right angle.

Definition: A ***Right Triangle*** is defined to be a triangle that has a right angle.

The ***hypotenuse*** of a right triangle is defined to be the side opposite the right angle.

The ***legs*** of a right triangle are defined to be the sides that are not the hypotenuse.

Theorem 6.4.2 (Perpendicular Distance Theorem)

If a line l is given and a point P is not on l, then the shortest distance $d(P, Q)$ among all points $Q \in l$ will be achieved by the unique point Q on l such that $\overleftrightarrow{PQ} \perp l$.

Definition of the Distance from a Point to a Line: Given a line l and a point P not on l, the **distance from *P* to *l*** is defined to be the distance PQ where Q is the unique point on l such that $\overleftrightarrow{PQ} \perp l$.

Definition of *Altitude Line, Foot of an Altitude Line, Altitude Segment*

An ***altitude line*** of a triangle is a line that passes through a vertex of the triangle and is perpendicular to the opposite side. (Note that the altitude line does not necessarily have to intersect the opposite side to be perpendicular to it. Also note that Corollary 6.3.4 tells us that there is exactly one altitude line for each vertex.) The point of intersection of the altitude line and the line determined by the opposite side is called the ***foot of the altitude***. An ***altitude segment*** has one endpoint at the vertex and the other endpoint at the foot of the altitude line drawn from that vertex.

Theorem 6.4.4 *Hypotenuse-Leg* (*HL*)

In Neutral Geometry, if there is a bijection between the vertices of any two right triangles, and the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

Theorem 6.4.6 In a neutral geometry, given a segment $\overline{AB}$ and a point P, the following are equivalent:

(i) P lies on the perpendicular bisector of $\overline{AB}$.

(ii) P is equidistant from the endpoints of the segment. That is, $PA = PB$.

Theorem 6.4.7 In a neutral geometry, given an angle $\angle ABC$ and a point $P \in \text{int}(\angle ABC)$, the following are equivalent:

(i) P lies on the bisector of $\angle ABC$.

(ii) P is equidistant from the rays of the angle. That is

the distance from P to line $\overleftrightarrow{BA}$ is equal to the distance from P to line $\overleftrightarrow{BC}$.

Definition of Circle

symbol: $\mathcal{C}_r(C)$

spoken: The **circle** with **center** C and **radius** r.

usage: C is a point in a metric geometry and r is a positive real number.

meaning: the set of points that are a distance r from point C. That is, $\mathcal{C}_r(C) = \{P \in \mathcal{P} | PC = r\}$

additional terminology: A **chord** of a circle is a segment $\overline{AB}$ with A, B distinct points on the circle.

A **diameter segment** of a circle is a chord that contains the center of the circle.

A **radius segment**, or **radial segment**, of a circle is a segment $\overline{CA}$ where C is the center of the circle and A is a point on the circle.

The **interior** of the circle $\mathcal{C}_r(C)$, denoted $\text{int}\{\mathcal{C}_r(C)\}$, is the set of points that are a distance less than r from point C. That is, $\text{int}\{\mathcal{C}_r(C)\} = \{P \in \mathcal{P} | PC < r\}$

The **exterior** of the circle $\mathcal{C}_r(C)$, denoted $\text{ext}\{\mathcal{C}_r(C)\}$, is the set of points that are a distance greater than r from point C. That is, $\text{ext}\{\mathcal{C}_r(C)\} = \{P \in \mathcal{P} | PC > r\}$

A **tangent line** for a circle is a line that intersects the circle at exactly one point.

A **secant line** for a circle is a line that intersects the circle at exactly two points.

Corollary 6.5.4 In a neutral geometry, the perpendicular bisector of any chord of a circle contains the center of the circle.

Theorem 6.5.5 In a neutral geometry, the interior of any circle is a convex set.

Theorem 6.5.6 In a neutral geometry, a tangent line for a circle is perpendicular to the radial segment at the point of tangency.

Theorem 6.5.7 (Existence and Uniqueness of Tangents)

In a neutral geometry, for every point on a circle, there is exactly one line that contains the given point and is tangent to the circle.

Theorem 6.5.9 (Line-Circle Theorem)

In a neutral geometry, if a line intersects the interior of a circle, then the line is a secant line.

Theorem 6.5.10 (Exterior Tangent Theorem)

In a neutral geometry, for every point in the exterior of a circle, there are exactly two lines that contain the given point and are tangent to the circle.