<u>Definitions and Theorems for Mon Feb 28 Quiz Q06 and Fri Mar 4 Exam X2</u>

2021 – 2022 Spring Semester MATH 3110/5110

Definition of Abstract Geometry (Barsamian's version, correcting an error in the book's definition)

An *abstract geometry* $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called *points* and $\mathcal{L}$ denotes a *non-empty* set whose elements are called *lines*, which are *sets of points* satisfying the following two requirements, called *axioms*:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in \mathcal{L}$ there exist **at least two** distinct points that are elements of the line.

Definition of Incidence Geometry

An *incidence geometry* is an ordered pair $(\mathcal{P}, \mathcal{L})$ that is satisfies all the requirements of an *abstract geometry* and also satisfies the following two additional *axioms*:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist **(at least) three** non-collinear points.

Definition of Notation for the Unique Line Containing Two Given Points

Symbol: $\overleftrightarrow{AB}$

Spoken: *line A B*

Usage: There is an *incidence geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ in the discussion and $A, B \in \mathcal{P}$ are two distinct points

Meaning: the unique line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,

If $l_1 \cap l_2$ has two or more distinct points,

then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

Corollary 2.1.7 (contrapositive of Theorem 2.1.6)

Given two lines l_1 and l_2 in an *incidence geometry*,

If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),

then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P,Q) \geq 0)$

(ii) $d(P,Q) = 0$ if and only if $P = Q$

(iii) $d(P,Q) = d(Q,P)$

Definition of a Ruler for a Line

words: f is a ruler for line l. (**alternate words:** f is a coordinate function for line l.)

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a *bijection*.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P,Q)$$

Additional Terminology:

The equation above is called the ***Ruler Equation***.

The number $f(P)$ is called the ***coordinate of P with respect to f***.

Definition of Metric Geometry

A *metric geometry* $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$.
- Every line $L \in \mathcal{L}$ has a *ruler*. (This requirement is called the ***Ruler Postulate***.)

Theorem 2.3.2 (Ruler Placement Theorem)

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- distinct points $A, B \in \mathcal{P}$

Claim: There exists a ruler g for line $\overleftrightarrow{AB}$ such that $g(A) = 0$ and $g(B) > 0$.

Terminology: Such a ruler g is called a ***ruler with A as origin and B positive***.

Definition of Betweenness for Real Numbers

Symbol: $x * y * z$

Spoken: *y is between x and z.*

Usage: $x, y, z \in \mathbb{R}$

Meaning: $x < y < z$ or $z < y < x$

Remark: It is a property of real numbers that for given any three distinct real numbers, one is smallest, one is largest, and the other is between them

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: *B is between A and C.*

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

Theorem 3.2.2 (Really a Corollary of the Definition)

Given: Points A, B, C in a metric geometry

Claim: The following are equivalent (*TFAE*): (i) $A - B - C$

(ii) $C - B - A$

Theorem 3.2.3 Betweenness of Points is Related to Betweenness of Coordinates

Given: Collinear points A, B, C on line l with ruler f in a metric geometry

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$ (betweenness of *points*)

(ii) $f(A) * f(B) * f(C)$ (betweenness of *coordinates*)

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

Claim: Exactly one of the points is between the other two.

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim: (i) There exists a point C with $A - B - C$

(ii) There exists a point D with $A - D - B$

Definition of Segment

Symbol: $\overline{AB}$

Spoken: *segment A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

Symbol: length$(\overline{AB})$

Spoken: the **length** of segment $\overline{AB}$

Meaning: the number AB. That is, the length is the number $d(A, B)$.

Definition of Ray

Symbol: $\overrightarrow{AB}$

Spoken: *ray A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\begin{aligned}\overrightarrow{AB} &= \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\} \\ &= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}\end{aligned}$$

Additional Terminology

The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

The **interior of the ray** is the set of all points of the ray except the initial point:

$$\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$

Theorem 3.3.4 Subtlety in the Notation for a Ray

(i) (Different symbols that represent the same ray.) If $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.

(ii) (If two rays are equal then their initial points are equal.) If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

Theorem: Existence and Uniqueness of the Midpoint of a Segment

If A, B are distinct points in a metric geometry, then segment $\overline{AB}$ has exactly one midpoint.

Definition of Angle

Symbol: $\angle ABC$

Spoken: *angle* $A\ B\ C$.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$

Additional Terminology: The **vertex** of $\angle ABC$ is the point B.

Definition of Triangle

Symbol: ΔABC

Spoken: *triangle* $A\ B\ C$.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

Additional Terminology:

The **vertices** of ΔABC are the points A, BC.

The **sides** (or **edges**) of ΔABC are the segments $\overline{AB}, \overline{BC}, \overline{CA}$.

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \ldots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. That is,

If $i \neq j$ then $A_i \cap A_j = \phi$

Definition of Convex

Words: S is convex

Usage: A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.

Meaning: for every two distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.

Quantified version: $\forall A, B \in S, A \neq B (\overline{AB} \subset S)$.

Universal Conditional Version: $\forall A, B \in \mathcal{P}, A \neq B$(If $A, B \in S$ then $\overline{AB} \subset S$)

Definition: The Plane Separation Axiom (PSA) (Barsamian's version of the definition)

Words: A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)

Meaning: For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

(i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

(ii) Each of the *half planes* is convex.

(iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.

Additional Terminology:

Line l is called the ***edge*** of *half planes* H_1 and H_2.

Words: *Points A, B lie on the* ***same side*** *of line l.*

Meaning: Points A, B are elements of the same half plane associated to l.

Words: *Points A, B lie on* ***opposite sides*** *of line l.*

Meaning: Points A, B are elements of different half planes associated to l.

PSA (ii) and (iii) and their Contrapositives

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.