1.2a: Introduction to Sets

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for Ohio University MATH 3110/5110 College Geometry

Subject: Introduction to Sets

- Definitions
- Proving a statement about subsets.

Textbook: Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition* (Springer, 1991, ISBN 3-540-97412-1)

Reading: Section 1.2 Sets and Equivalence Relations, pages 3 – 4

Homework: Section 1.2 # 1,3

Remarks on Reading the Book

The book that we are using for this course is an advanced book. Many (maybe most) of you will have difficulty reading the book, especially early in the semester. In particular, the book assumes that you have some familiarity with

- Logical Terminology and Notation.
- General Mathematical Conventions
- Reading Proofs
- Writing Proofs

For my part, I will be discussing some of the above concepts in meetings and in videos, in order to help you come up to speed.

For your part, you will need to work hard to get to a higher level of math reading skill. In particular, you must get used to the idea that when reading dense technical writing, you may have to read the same material multiple times. Each time you read it, you will learn something new.

Part 1: Introduction to Sets

Section 1.2 begins with some defintions of basic set terminology on page 4.

Definition of *Subset*

Symbol: $T \subset S$

Spoken: *T is a subset of S*

Meaning In Words: Every element of T is also an element of S.

With Universal Quantifier more explicitly stated:

For every x in T, x is an element of S.

Expressed as a Universal Conditional Statement:

For every x in the Universal Set, if x is an element of T then x an element of S.

Meaning abbreviated in symbols:

Written as a Universal Statement: $\forall x \in T(x \in S)$

Written as a Universal Conditional Statement: $\forall x \in U(\text{If } x \in T \text{ then } x \in S)$

Remark: Some books use the symbol $T \subseteq S$ for subset.

Set Equality

Symbol: $T = S$

Spoken: T equals S

Usage: T and S are sets

Meaning: Sets T and S have the same elements. Another way of saying this is that regardless of the element x chosen from the universal set U, the statements $x \in T$ and $x \in S$ will both be true or they will both be false.

Meaning stated using Subset Notation: $T \subset S$ and $S \subset T$

The Empty Set

Symbol: ϕ

Spoken: the empty set

Meaning: The set with no elements.

Remark: It is a fact that the empty set ϕ is a subset of every set. That is, $\forall S(\phi \subset S)$. This fact is proven in courses like MATH 3050.

Operations on Sets

Symbol: $A \cup B$

Spoken: *The union of A and B*

Meaning: $\{x \in U | x \in A \text{ or } x \in B\}$

Symbol: $A \cap B$

Spoken: *The intersection of A and B*

Meaning: $\{x \in U | x \in A \text{ and } x \in B\}$

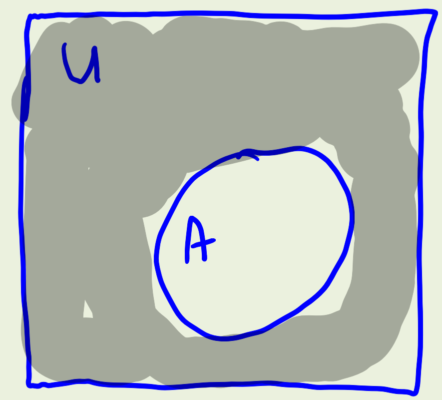
Words: Sets A and B are *disjoint*.

Meaning: $A \cap B = \phi$

Symbol: A^c

Spoken: *The complement of A in U*.

Meaning: $\{x \in U | x \notin A\}$

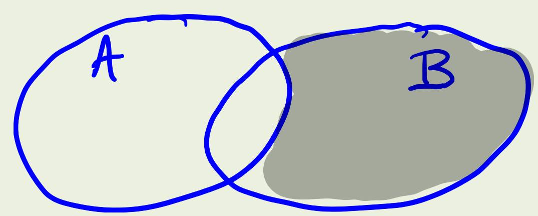


Symbol: $B - A$

Spoken: *B minus A*

Meaning: $\{x \in U | x \in B \text{ and } x \notin A\}$

More abbreviated meaning: $B \cap A^c$



[Example 1] (Similar to 1.2#1) Prove that if $A \subset B$ then $A \cup C \subset B \cup C$.

Solution Recall from MATH 3050 (or CS 3000) four things:

(1) The statement above is *implicitly quantified*. That is, the statement is claiming to be true for all sets A, B, C. We can write the quantifier explicitly:

For all sets A, B, C, if $A \subset B$ then $A \cup C \subset B \cup C$.

(2) The *For all* quantifier is called the *Universal Quantifier*. It can be abbreviated

$$\forall sets\ A, B, C (If\ A \subset B\ then\ A \cup C \subset B \cup C)$$

(3) The form of a *Conditional Statement*: $If\ P\ then\ Q$.

Statement P is called the hypothesis; Statement Q is called the *conclusion*.

A conditional statement with a universal quantifier is called a *universal conditional statement*.

(4) A universal conditional statement can be proven with a Direct Proof. That is, the form:

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

---------- some steps here ------------

(*) Therefore $A \cup C \subset B \cup C$ is true. (some justification)

End of Proof

So far, we have a frame of a proof. That is, we know how it must begin and end.

Now, notice that the first of the proof contains the term $\subset$, which is spoken *is a subset of*. This is a *defined term*. (The definition is on page 4 of the book.) The only thing that can follow statement (1) is a new statement that expresses what the subset symbol really means. That is, we have no choice about statement (2).

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*). unpack the definition

---------- some steps here ------------

(*) Therefore $A \cup C \subset B \cup C$ is true. (some justification)

End of Proof

What we have done in going from step (1) to step (2) is that we have replaced the abbreviated symbol $A \subset B$ with the less abbreviated sentence

For all x, if $x \in A$ then $x \in B$.

by using the definition of *subset*. This is called *unpacking the definition*. What we have done is an example of the following general practice in writing proofs:

> *When one is given a statement that involves a defined expression, one must follow that statement with a new statement that is the unpacked version of the defined expression.*

Realize now that the final statement (*) also contains the defined term $\subset$. The only way to get to that statement is to have preceeding it another statement that expresses what (*) means, but in unabbreviated form.

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

---------- some steps here ------------

(**) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (some justification) Pack up the definition

(*) Therefore $A \cup C \subset B \cup C$ is true. (**by ~~(*)~~ (**) and definition of subset**)

End of Proof

What we have done in going from step (**) to step (*) is that we have replaced a non-abbreviated statement with an abbreviated statement, using the definition of *subset*. This is called *packing up the definition.* What we have done is an example of the following general practice:

> *To prove a statement that involves a defined expression (that is, an abbreviated expression), one must first prove a statement that is not abbreviated.*

Notice that we have no choice about the 2nd to last statement of the proof. And furthermore, we have no choice about how the final statement of the proof must be justified.

Realize that statement (**) is a Universal Conditional Statement. It can be proven by a little *Direct Proof* of its own. That is, a sequence of steps like the following.

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

frame of direct proof for statement (**)

(3) Suppose that y is an element such that $y \in A \cup C$

---------- some steps here ------------

(***) Therefore $y \in B \cup C$. (some justification)

(**) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (**by steps (3) through (***)**)

(*) Therefore $A \cup C \subset B \cup C$ is true. (by (*) and definition of subset)

End of Proof

Now realize that statement (3), which we know is true, contains a defined expression, $A \cup C$. We have no choice about what must follow statement (3): we must *unpack* that definition.

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

(3) Suppose that y is an element such that $y \in A \cup C$

unpack the definition of subset

(4) $\boldsymbol{y \in A}$ or $\boldsymbol{y \in C}$ (by (3) and definition of union)

---------- some steps here ------------

(***) Therefore $y \in B \cup C$. (some justification)

(**) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (by steps (3) through (***))

(*) Therefore $A \cup C \subset B \cup C$ is true. (by (*) and definition of subset)

End of Proof

Also realize that the statement that we need to prove, statement (***), contains a defined expression, $B \cup C$. We have no choice about what statement must come before statement (***): We must prove the statement of what (***) really means. Then statement (***) will follow by the definition of union. (That is, we will *pack up* the definition.)

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

(3) Suppose that y is an element such that $y \in A \cup C$

(4) $y \in A$ or $y \in C$ (by (3) and definition of union)

---------- some steps here ------------

(****) $\boldsymbol{y \in B}$ or $\boldsymbol{y \in C}$ (some justification)

Pack up the definition of union

(***) Therefore $y \in B \cup C$. **(by (****) and definition of union)**

(**) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (by steps (3) through (***))

(*) Therefore $A \cup C \subset B \cup C$ is true. (by (*) and definition of subset)

End of Proof

Let's pause and reflect on what we have done so far. We have assembled eight statements of the proof without really having any choice about them. That is idea of what I refer to as ***considering proof structure***. For many students, the idea of writing proofs is rather daunting, with some mystery about how proofs are conjured up. It is true that there is sometimes a leap of inspiration that is necessary to complete a proof. But often, much of what happens in a proof is *not* mysterious. Some steps are inevitable. In particular, the frame of the proof—the beginning and end—are often very formulaic. The result is that the mysterious part of the proof—the gap that you must leap—can be narrowed by *considering proof structure*.

In our proof, the gap that we must leap is from Statement (4) to Statement (****). Notice that Statement (4) is an OR statement. That gives us a clue that the way to proceed is to use *proof by cases*.

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

(3) Suppose that y is an element such that $y \in A \cup C$

(4) $y \in A$ or $y \in C$ (by (3) and definition of union)

(5) (Case I) Suppose $y \in A$ (assumption for proof by cases)

(6) then $y \in B$ (by (5) and (1))

(7) So $y \in B$ or $y \in C$ in this case (~~**(by (5))**~~ by (6))

(8) (Case II) Suppose $y \in C$ (assumption for proof by cases)

(9) So $y \in B$ or $y \in C$ in this case (by (8))

(****) **(Conclusion of Cases)** $y \in B$ or $y \in C$ (**by (4),(7),(9) and method of proof by cases**)

(***) Therefore $y \in B \cup C$. (by (****) and definition of union)

(**) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (by steps (3) through (***))

(*) Therefore $A \cup C \subset B \cup C$ is true. (by (*) and definition of subset)

End of Proof

With that, we realize that we have closed the gap, and we can number our statements.

Proof (Direct Proof)

(1) Suppose that A, B, C are sets such that $A \subset B$ is true.

(2) For all x, if $x \in A$ then $x \in B$. (by (1) and definition of *subset*).

(3) Suppose that y is an element such that $y \in A \cup C$

(4) $y \in A$ or $y \in C$ (by (3) and definition of union)

(5) (Case I) Suppose $y \in A$ (assumption for proof by cases)

(6) then $y \in B$ (by (5) and (1))

(7) So $y \in B$ or $y \in C$ in this case (by (5))

(8) (Case II) Suppose $y \in C$ (assumption for proof by cases)

(9) So $y \in B$ or $y \in C$ in this case (by (8))

(10) (Conclusion of Cases) $y \in B$ or $y \in C$ (by (4),(7),(9), and method of proof by cases)

(11) Therefore $y \in B \cup C$. (by (10) and definition of union)

(12) For all y, if $y \in A \cup C$ then $y \in B \cup C$ is true. (by steps (3) through (11))

(13) Therefore $A \cup C \subset B \cup C$ is true. (by (12) and definition of subset)

End of Proof

End of [Example 1] and End of Video